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AZIMUTH

CRAIG

AZIMUTH.

A TREATISE ON THIS SUBJECT,

WITH A STUDY OF THE ASTRONOMICAL TRIANGLE, AND OF THE EFFECT OF ERRORS IN THE DATA.

ILLUSTRATED BY LOCI OF MAXIMUM AND MINIMUM ERRORS.

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AZIMUTH.

PART I.

INTRODUCTION.

1. It is purposed in the following notes to suggest a study of the astronomical triangle with respect to the azimuth problem, supplementing the teaching of the text-books; also, to point out false teaching of the latter on some points referring to the most favorable conditions of observation to reduce to a minimum in the computed azimuth the effect of small errors in the *data*.

A thorough study of the astronomical triangle, viewing it in all possible aspects, in the several problems of nautical astronomy—time, latitude, and azimuth—is instructive as an exercise, even when investigation is extended beyond the restricting limits imposed by the practical problem; that is, if free to select any part of the triangle, to take successively each part as fixed, and then to seek the best condition for all the remaining parts; one of these predominating in influencing a choice, and so making a third part subservient to fixing the others.

2. The problem presented in practice restricts the observer to a particular spot, and many times the observer limits himself to the observation of a particular celestial body: then the latitude and the declination are fixed, and it remains to find the value of some other one part affording the most favorable condition for the observation of the body when considering the error in each *datum* taken separately; and, finally, by exercise of the judgment, to choose the condition giving the best result when all the errrors in the *data* are considered. In practice, to know when to observe, the most convenient third part to find is either the hour-angle or the altitude; primarily the former, but, for discussion, sometimes the latter; from which is easily found the hour-angle to give the time for the observation.

If not restricted to the employment of a particular body, the first step will be to select the best one from several bodies that are favorably situated for observation.

3. Since the azimuth of a heavenly body is determined for the purpose of ascertaining (1) at sea the compass error, thence the variation of the magnetic needle or the deviation caused by the iron in the ship, or both variation and deviation; (2) on board ship, the true bearing of some point visible on land; or (3) on land the same, thence the direction of the meridian—the best practical conditions should be sought.

The lower the altitude the better for an observation of the compass azimuth; an altitude of the heavenly body equal to that of the object on land being the best for the determination of the horizontal angle between the two objects by means of the theodolite.

or for the observation of an astronomical bearing (sextant being used) from which the horizontal angle is to be computed: because then the error of level will cause the least effect in observations with the compass and the theodolite, and the error in the observed sextant-angle will be multiplied less the more nearly the altitudes agree.

Since the terrestrial object should be near the horizon, we may say, in general, the lower the altitude the better.

On the other hand, if altitude enters the data, the lower the altitude the more uncertain the refraction, and the computed azimuth will be correspondingly affected by error in altitude. In this case, then, very low altitudes, and in all cases very high altitudes, should be avoided, whatever conditions are given from the astronomical triangle as theoretically the most favorable for precision in the computed azimuth. Intelligent discrimination should be exercised by the observer to effect a compromise between the theoretical and the practical advantages.

4. In ordinary circumstances at sea it is unnecessary to exact the best condition when great labor would attend ascertaining it; as, for instance, in the case of a body that crosses the prime vertical above the horizon, to find the best position in which to observe it, to make the effect of a small error in altitude the least; for to do this requires the solution of a trigonometric equation of the fourth degree. Nevertheless, very bad conditions should be avoided.

In the case of serial time-azimuths to determine the deviation of the magnetic needle, considerable work may properly be demanded to ensure taking observations extending over the most favorable time, on each side; for, granting the impossibility of obtaining very nice results from observations made at sea, yet as good work as is practicable should be done, reducing the errors as much as possible.

In a survey on land, with so much better conditions, errors may be eliminated to a great degree; and the nearest approach to accuracy that can be made should be insisted on, even though much labor attends the finding of the best conditions for observation.

5. The truth should be known, notwithstanding the knowledge of it may not always be put to effective use; therefore erroneous assertions in well-known and much-used text-books should not be perpetuated.

In the following extracts the italics are the writer's, not the authors':

MAYNE'S MARINE SURVEYING, page 90, treating of true bearings, says: "Firstly, the body should be rising or falling rapidly, when its movement in azimuth will also be *rapid*." *

In the first place, *slow* should be substituted for *rapid;* but, even with this correction made, the words in italics are not true in reference to any body not on the equator † that, in its diurnal course, crosses the prime vertical. Though the author favors the method of time-azimuth, he makes his remarks apply to the altitude-azimuth as well. Now, it is not the

* "In other words, *the nearer the prime vertical the better.*"

† If the declination of the body is zero, the most favorable position, theoretically speaking, is when on the prime vertical—that is, at the intersection of the horizon, equator, and prime vertical ($t = 6^h$, $h = 0$, $Z = 90°$).

This one exception must be *understood* throughout, in the statements denying that the position of the body when *crossing* the prime vertical is the most favorable and asserting that on the six-hour angle is a better position. In *this* case $t = 90°$ and $Z = 90°$ coexist and give the best position. But, if understood, it will not be necessary to repeat that this single exception exists.

When $d = L$, the body *not crossing* is in the best position, theoretically, when *on* the prime vertical.

ratio existing between change of altitude and change of time that should be considered; but, in the case of time-azimuth, the ratio of change of time to change of bearing, and, in altitude-azimuth, the ratio of change of altitude to change of bearing. With either method employed, so far as error in time or error in altitude is concerned, the body will be more favorably situated at the point of crossing the six-hour circle than at transit on the prime vertical (see articles 43 and 45). But the best position lies between these two points, in the case of altitude-azimuth for all latitudes; and in time-azimuth for all latitudes less than 45°, while with increasing latitudes the best position for certain declinations approaches the meridian in bearing, and finally reaches it (see art. 95).

While considerable labor may be required to determine this position in order to seize the observation there, it can be taken at the first point mentioned ($t = 90$) with less trouble than if observed at the second ($Z = 90$), often with the further advantage of having a better altitude for observing the compass-azimuth and the astronomical bearing of the object on the earth. This is worth knowing in cases where the labor of finding the best position may reasonably be dispensed with.

6. CHAUVENET'S ASTRONOMY, vol. i., page 431, treating of true bearings, employing the method of altitude-azimuth, says: "From the first equation of (50), ϕ and ϑ being constant, $dA = -\dfrac{dh}{\cos h \tan q}$, *and therefore an error in the observed altitude will have the least effect upon the computed azimuth when tan q is a maximum; that is, when the star is on the prime vertical. Therefore, in the practice of the preceding method the star should be as far from the meridian as possible*" [far, in bearing or azimuth].

For a star that crosses the prime vertical the quotation in italics is not true. Instead of "*when tan q is a maximum,*" read *when tan q cos h is a maximum*, and reject all that follows; and this without disputing the correctness of the language intended to convey the meaning.

7. But it may be pertinent to remark here that text-writers have a careless habit of using the words *far from* and *near to*, referring to the meridian and the prime vertical. They know what they mean to say, but may mislead the novice. The primary idea conveyed by this language is *distance*, absolute, which in this particular case would place the body on the six-hour circle, while in the p. v. is meant; the secondary idea, that of *time-measurement* from the upper branch of the meridian, which would place the body in the horizon; and, last of all, the idea of *angular* distance in azimuth (or bearing), yet this is what is meant. Say *far from* (*or near*) *in azimuth, or in bearing;* or else say *when the body bears most nearly east and west, or north and south,* as the case may be. As an instance of carelessness, a standard work—and many others err in the same way—states that for an error in altitude in the time-sight the effect on the computed time will be least when the body is observed *nearest* the prime vertical. Now, a body whose declination is greater than the latitude of the same name will be nearest the prime vertical when on the meridian, at upper culmination, when the zenith distance is least; and this is the worst possible position when considering errors in altitude not only, but in latitude and declination as well. Growing out of this carelessness on the part of intelligent authors, many text-books on navigation, compiled by writers without thinking, publish the absurd statement that the method of single-altitude observation for latitude should be used only when the heavenly body is within an hour (or thereabouts) of the meridian. Believing this, what would become of the reputation of the

pole-star, earned by its unvarying good character for a latitude-observation throughout its diurnal course?

8. COFFIN'S NAVIGATION AND NAUTICAL ASTRONOMY, Fifth Edition, page 277, treating of true bearings, says: "Z_s = NZM, the azimuth of the celestial body, may be found from an observed altitude (Prob. 40), or from the local time (Prob. 38).

"*In the first case the most favorable position is on or nearest the prime vertical; for then the azimuth changes most slowly with the altitude.* In the latter, positions near the meridian may also be successfully used."

For a body that crosses the prime vertical the italicized statement is erroneous. While the last part of the quotation (not in italics), in respect to error in time, is true; yet the word *also* implies that the best position is on the prime vertical, which is not true. It is worth remarking in this case (time-azimuth) that an error in latitude will produce a maximum effect in the computed azimuth if the body is observed at a particular point lying between the intersections of the prime vertical and the meridian with the diurnal circle; and this point should be avoided. Dependent on the relative values of the latitude and declination, it may be, in bearing, far from or near to the meridian.

9. MARINE SURVEYING, U. S. NAVAL ACADEMY, page 25, on the subject of true bearings, treating of time-azimuth, to be superseded by altitude-azimuth if the time is not accurately known (page 26), says: "The observation should be made when the heavenly body *is near the prime vertical*, provided it has not too great an altitude at that time."

10. Even if *on the prime vertical* were the best position considering error in altitude alone or time alone, there is another datum—the latitude—that may have a considerable error, while both the time and altitude may be determined to a comparatively great degree of accuracy. In this case if the method of altitude-azimuth is chosen, a small error in latitude will produce no error in the computed azimuth, provided the body is observed when on the six-hour circle; hence, this is the most favorable position for error in latitude, and it is a better position than that on the prime vertical for error in altitude.

If time-azimuth is used, a rising-and-setting body will be in the best position, respecting error in latitude, when in the horizon—the effect of this error then reducing to zero. And this position may be better than that on the prime vertical for error in time—depending on the relative values of the latitude and declination.

11. Not unfrequently in surveying, a true bearing may be demanded for preliminary plotting before either time or latitude has been accurately determined. The latter, as well as the longitude, will in many cases be known approximately: an error in the assumed latitude then need not deter the surveyor from using the altitude-azimuth, with excellent results, when the body is on or near the six-hour circle.

Circumstances may have foiled the observer's attempts to determine the latitude, and yet have permitted excellent observations of equal-altitudes for time. He then can choose between the two methods for azimuth; or a third method, the time-altitude-azimuth, in which latitude does not enter as a part, may be employed, provided the altitude is not very great when the body is observed near the meridian in bearing, and the hour-angle is not very small. This method may often be employed with excellent results in observations of the sun when the declination and the latitude have contrary names (see art. 14), and still better when a close circumpolar star is observed.

12. If, at a particular instant, an observation for azimuth is wanted, and the time is accurately known, the first thought will be to use the convenient time-azimuth tables; or, if the time is not well known, to employ the method of altitude-azimuth. But supposing both of these data accurately known and the latitude uncertain, which of the two methods shall be chosen?

Even admitting comparatively slight errors in altitude and time, which shall be the choice?

If the last-mentioned errors should have exactly the same value (measured by the same unit, in arc), and considering any error existing in the declination, but that no error in the latitude exists, the method of time-azimuth would be the better for the observation taken at any time whatever,—excepting at the single instant when the parallactic angle might have the value of ninety degrees ($q = 90°$), possible only when the declination exceeds the latitude; for then the two methods would give identical values to the total error in the computed azimuth. Therefore, if the latitude could always be known exactly, and time and altitude were equally trustworthy, the choice should always fall on the time-azimuth.

13. Admitting the uncertain and considerable error in latitude, while the conditions of the preceding paragraph remain otherwise unchanged, the effect of this error alone must be the criterion for choosing the method to use.

For a circumpolar star whose declination is greater than the latitude, the time-azimuth will be the better from the time of lower culmination until a point e is reached by the star before it arrives on the six-hour circle ($t = 90°$); the altitude-azimuth will be better from this point e until the star attains its greatest elongation ($q = 90°$); thence, until its upper culmination, the time-azimuth the better.* If not a circumpolar star, the time-azimuth is better from the time of rising until at e' before reaching $t = 90°$; thenceforward to the meridian the same as for a circumpolar star.

If the declination is less than the latitude of the same name, by substituting "reaches the prime vertical ($Z = 90°$)" for "attains its greatest elongation ($q = 90°$)," the foregoing statement will apply.

If the declination is zero, or has the contrary name to that of the latitude, the time-azimuth will be the more favorable during the whole time *of visibility* of the star.

14. The method of time-altitude-azimuth, though generally ignored by navigators, is deserving of attention, as giving advantage in some circumstances over both the time-azimuth, and the altitude-azimuth. It should not be used when Z is near 90°, t near 0°, or with a high altitude (see art. 11); but at sea, when the ship and the sun are on opposite sides of the equator, there is a wide extent of cruising-space where, at the best, poor conditions are given for observations of the sun for time. The azimuth determined by this third method, from the same altitude observed for time, is more conveniently found than if taken from the time-azimuth tables, considering the double interpolation needed for the actual latitude and declination. For, working side by side with the computation for time, at the same opening of the logarithmic tables required by the latter, only two additional logarithms are needed together with the arithmetical complement of one already taken out, the sum of the three giving

* Considering the eastern hemisphere alone, since symmetrical conditions exist west of the meridian, and limiting the angles to 180°, as in art. 17.

the log sine of the whole azimuth. The chief error will be owing to the error in the computed time, arising from any existing error in the latitude; but the azimuth taken from the tables will embrace this error, and the error in the latitude itself, the latter not entering in the method under discussion. Recourse may be had to the altitude-azimuth, but here the error in latitude enters, and to a much greater degree than in the time-azimuth; while the work of computation is considerably greater than in the time-altitude-azimuth. By the latter, the azimuth being derived from its sine, two values will be given, greater and less, respectively, than 90°; but the conditions of observation readily determine the actual value.

15. The method of horizon-azimuth or amplitude, while useful at sea, and very convenient in the case of the sun on account of the prepared tables, should not be used in nice work; since the degree of correctness in the result depends on the approach to accuracy in *estimating* that the body, whose image is lifted by uncertain refraction, is in the true horizon. Rejecting this fourth method in the summing-up, and noting that the single case of declination equal to zero is excepted from the truth of the statements, we have the following:

SUMMARY.

1st. As to errors in declination and in latitude, the body on the prime vertical is not in its most favorable position for reducing the error in the azimuth computed by any one of the three methods.

2d. If a body crosses the prime vertical, the best position is not on it for the error in any single datum in any one of the three methods.

3d. For error in t in the time-azimuth or for error in h in the altitude-azimuth, *the body that crosses the prime vertical is in a less favorable position when on the prime vertical than when on the six-hour circle; but the latter is not the best position.* The best will not, however, be the same point for error in h that it is for error in t.

PART II.

DEDUCTION OF FORMULAS FOR THE DETERMINATION OF AZIMUTH.

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16. Thus far assertions alone have been given; their truth remains to be proved. The notation used will be as follows:

d,	for the	declination of the body;				
h,	" "	altitude	" " "			
t,	" "	hour angle	" " "			
Z,	" "	azimuth	" " "			
q,	" "	parallactic angle of the body;				
p, or *co-d*,	" "	polar distance of the body,		$= (90^\circ - d)$,	observing sign;	
z, or *co-h*,	" "	zenith distance	" " "	$= (90^\circ - h)$,	"	"
L,	" "	latitude	" " place;			
co-L,	" "	co-latitude	" " "	$= (90^\circ - L)$.		

17. In *computing* the azimuth the *hour-angle t* will be (see art. 34) regarded as positive from 0 at the upper culmination of the body to 180° at the lower culmination, whether to the west or east from the meridian; the *azimuth Z* as positive from 0°, at the point of the horizon nearest to the elevated pole (*N.* or *S.* point) around to 180° at the opposite point of the horizon (*S.* or *N.* point), whether to the west or to the east; the *parallactic angle q* as always positive, 0° up to a possible 180°; the *latitude L* as always positive towards the elevated pole, from 0° to 90°; the *declination d* as positive if of the same name as the latitude, as negative if of contrary name, 0° to ± 90°; the *polar distance p* as always positive from 0° at the elevated pole to 180° at the depressed pole; the *zenith distance z* as always positive from 0° at the zenith to 180° at the nadir; the *altitude h* as positive if above the horizon, negative if below, 0° to ± 90°: the *co-latitude co-L* as always positive, 0° to 90°.

18. The following well-known formulas are given for convenience of reference:

ALTITUDE-AZIMUTH.

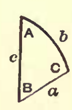

FIG. 1.

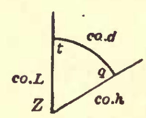

FIG. 2.

Given h, d, and L, to find Z.

From the fundamental formula of trigonometry,

$$\left.\begin{aligned}\cos b &= \cos a \cos c + \sin a \sin c \cos B, \text{ we have}\\ \sin d &= \sin h \sin L + \cos h \cos L \cos Z\end{aligned}\right\} \quad (1)$$

$$\cos Z = \frac{\sin d - \sin h \sin L}{\cos h \cos L}; \quad (2)$$

from which are derived

$$\cos \tfrac{1}{2}Z = \sqrt{\frac{\cos s \cos (s-p)}{\cos L \cos h}} \quad \text{when} \quad s = \tfrac{1}{2}(L+h+p); \quad . \; . \; . \; . \quad (3)$$

$$\sin \tfrac{1}{2}Z = \sqrt{\frac{\cos s \sin (s-d)}{\sin co\text{-}L \cos h}} \quad \text{when} \quad s = \tfrac{1}{2}(co\text{-}L+h+d); \quad . \; . \; . \quad (4)$$

$$\tan \tfrac{1}{2}Z = \sqrt{\frac{\sin (s-L) \sin (s-h)}{\cos s \cos (s-p)}} \quad \text{when} \quad s = \tfrac{1}{2}(L+h+p). \; . \; . \; . \quad (5)$$

Small errors in the data will have the same effect on the computed azimuth, whichever formula is used. So far as inexactness in the logarithmic tables is concerned, formula (5) will give the result nearest to precision, since the tangent of an angle varies more rapidly than either the sine or the cosine.

If $Z > 90°$, (3) will be better than (4); if $Z < 90°$, (4) is preferable to (3); since the cosine varies more rapidly than the sine for an angle greater than 45°, and less rapidly for an angle less than 45°.

19. TIME-AZIMUTH.

Given t, L, and d, to find Z.

From the fundamental formula, we have

$$\left.\begin{aligned} \sin A \cot B &= \sin c \cot b - \cos c \cos A; \\ \sin t \cot Z &= \cos L \tan d - \sin L \cos t; \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

$$\cot Z = \frac{\cos L \tan d - \sin L \cos t}{\sin t}; \quad . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

from which are derived

$$\tan \phi = \cot d \cos t; \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

$$\cot Z = \frac{\cos (\phi + L) \cot t}{\sin \phi}; \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

$$\sin h = \frac{\sin (\phi + L) \sin d}{\cos \phi}. \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (10)$$

For the solution of an astronomical bearing—the altitude of the heavenly body not being observed—formula (10) will be needed for finding the true altitude, from which the required apparent altitude will be determined.

It will be convenient to accept the value of ϕ less than 90°, taking care to give it the proper sign. There will be no ambiguity in the value of Z found from its cotangent. If the latter is positive, the azimuth is less than 90°; if negative, greater than 90°.

The following formulas, derived from (7), are given in some text-books to the *exclusion* of the preceding:

$$S = \tfrac{1}{2}(p + co\text{-}L)\,; \quad \ldots \quad (11)$$
$$D = \tfrac{1}{2}(p \sim co\text{-}L), \text{ the greater minus the less}; \quad \ldots \quad (12)$$
$$\tan X = \sin D \operatorname{cosec} S \cot \tfrac{1}{2}t\,; \quad \ldots \quad (13)$$
$$\tan Y = \cos D \sec S \cot \tfrac{1}{2}t\,; \quad \ldots \quad (14)$$
$$Z = X \pm Y. \quad \ldots \quad (15)$$

Case I.—When $S < 90°$:

$$\text{If } p > coL,\ X + Y = Z. \quad \ldots \quad (16)$$
$$\text{If } p < coL,\ X - Y = Z. \quad \ldots \quad (17)$$

Case II.—When $S > 90°$:

$$180° - (X - Y) = Z. \quad \ldots \quad (18)$$

These formulas are convenient when a series of observations is to be taken, as in the case of serial time-azimuths, therefore convenient for preparing a table of azimuths; because *sin D, cos D, cosec S,* and *sec S* have constant logarithms (in the case of the sun regarding the declination at the mean of the times of observation as constant); but for a single observation they give more labor than (8) and (9) exact, and some trouble in freeing the result from ambiguity, whereas (9) is perfectly clear.

If, however, besides Z the angle q were required to be computed, formulas (11) to (14) would be the most convenient.

If $p > coL$, for (12), write $D = \frac{1}{2}(p - coL)$;

$$\left.\begin{array}{l}\text{for } \tan X, \text{ write } \tan \frac{1}{2}(Z - q) \\ \text{for } \tan Y, \text{ write } \tan \frac{1}{2}(Z + q)\end{array}\right\}\begin{array}{l}\text{whence sum} = Z; \\ \text{whence difference} = q.\end{array}$$

If $p < coL$,

for D, write $(coL - p)$;

$$\left.\begin{array}{l}\text{for } \tan X, \text{ write } \tan \frac{1}{2}(q - Z), \\ \text{for } \tan Y, \text{ write } \tan \frac{1}{2}(q + Z),\end{array}\right\}\begin{array}{l}\text{whence sum gives } q; \\ \text{and difference gives } Z.\end{array}$$

20. TIME-ALTITUDE-AZIMUTH.

Given t, h, and d, to find Z:

$$\left.\begin{array}{l}\sin a \sin B = \sin b \sin A \\ \cos h \sin Z = \cos d \sin t\end{array}\right\} \quad \ldots \quad (19)$$

$$\sin Z = \frac{\sin t \cos d}{\cos h}. \quad \ldots \quad (20)$$

Of the two values of Z given by (20), supplements of each other, the proper one may be known from the conditions of the case. If Z is very near $90°$, and it is doubtful on which side of the prime vertical the body lies, the ambiguity may not be removed; but the method is inadmissible then, since a small error in any given part may make Z impossible, sin Z being greater than unity; and in any event such error will produce in Z a very large error (see art. 14).

21. HORIZON-AZIMUTH.

Given $h = 0$, d, and L, to find Z:

From (2),
$$\cos Z = \frac{\sin d}{\cos L}. \qquad (21)$$

Attending to the sign of d, no ambiguity can arise.

22. A fifth method, which the writer has never seen alluded to, may be employed. It may be termed TIME-ALTITUDE-LATITUDE-AZIMUTH, to distinguish it from that of art. 20, which, to be precise, should be called Time-altitude-declination-azimuth.

Given t, h, and L, to find Z. This method may be resorted to in the remote contingency of declination not known, as, for instance, the mutilation of the ephemeris so far as concerns the declination only.

From trig.,
$$\left.\begin{aligned} \sin B \cot A &= \sin c \cot a - \cos c \cos B; \\ \sin Z \cot t &= \cos L \tan h - \sin L \cos Z; \end{aligned}\right\} \qquad (22)$$

whence
$$\left.\begin{aligned} \cot \vartheta &= \tan A \cos c; \\ \cos \vartheta' &= \cos \vartheta \tan c \cot a; \\ B &= \vartheta \pm \vartheta'; \end{aligned}\right\} \qquad (23)$$

and
$$\left.\begin{aligned} \cot \vartheta &= \sin L \tan t; \\ \cos \vartheta' &= \cos \vartheta \cot L \tan h; \\ Z &= \vartheta \pm \vartheta'. \end{aligned}\right\} \qquad (2$$

Attending to the signs, there will still be two values found for Z. But, since negative values of Z and values greater than $180°$ are excluded (art. 17), the proper value will be known.

When d is given it will be more trustworthy than is usually the L, hence this method is not considered farther, though having a place in the problem of azimuths.

PART III.

DIFFERENTIAL VARIATIONS IN THE ASTRONOMICAL TRIANGLE WITH REFERENCE TO AZIMUTH.

23. The *error in the computed azimuth arising from small errors in the data* may be translated into *the change in the value of the azimuth corresponding to small changes in the data.*

Since with three parts given in the *astronomical triangle*—and not less than three parts—the remaining parts can be found, we have in the problem of *azimuths* four parts in each of the methods proposed.

In order that the astronomical triangle shall change its aspect, not more than two parts can remain constant—all the others must vary.*

In the fundamental formulas employed, differentiating for any two parts of the triangle as variable, the other two being constant, the change in one variable may be found in some terms of the other variable and the two constants, and in a simpler form, sometimes, by employing parts of the triangle that do not enter the problem. For the investigation of maximum and minimum effects of errors, these equivalent expressions may be used with advantage.

24. In this problem, since Z is regarded always as one of the variables, we shall find the expression for the approximate error in Z corresponding to a small error in each of the given parts, taken separately, the other two parts being regarded as constant for that occasion.

The expression for the total error in Z will be, for practical purposes, the algebraic sum of the three errors thus found; that is, the total differential equals the sum of the partial derivatives.

25. In each case the expression for error depends on the particular parts of the triangle used, both the variables and the two constants; the remaining two parts, though they do not enter the formula, vary with the variables employed.

For illustration, the error in the computed azimuth owing to a small error in latitude will not have the same value when the method of time-azimuth is employed that it will have in the altitude-azimuth computation—excepting as stated in art. 13, for one point of observation of the celestial body; namely, when either q or Z equals 90°, whichever may be possible; at e or e' the numerical values of the resulting errors will be the same by both methods, but the signs will be contrary. Therefore, in practice, having determined the azimuth, and afterwards finding that the latitude used is slightly erroneous, care must be taken in correcting the result that dZ is not computed from the expression derived from the formula of the method *not used* to compute the azimuth. The foregoing applies as well to the case of error in declination; but in practice the occasions will be few when the declination can be regarded as having any appreciable error. For discussion, however, a sensible error in declination affords interesting study, and it will not be omitted in showing the relations of the errors in the data.

Though the expressions for the errors are not strictly true excepting when the incre-

* Except in the one peculiar case of two sides and the angles opposite them remaining constantly equal, each to 90°; while the remaining side and its opposite angle are the only variables, varying by the same amount, being always equal each to the other, in angular measure.

ments of change are infinitely small, yet they are nearly enough true provided the change in circular measure is sufficiently small to be regarded equal to its sine ; therefore, since an angle that does not exceed one degree may be regarded equal to its sine to insure accuracy to the nearest second in the result, the small errors in the data usually met *in practice* may be dealt with. dZ may, however, fall beyond the limits allowable as a finite difference for numerical use as a correction.

26. The increment of change is really length in arc. If given in seconds or minutes it may readily be reduced to circular measure, the radius being unity; but, in practice, this need not be done, for, the error sought being expressed in the same unit as the given error (in seconds or minutes), the factor for reduction common to both will divide out.

27. The expressions are often useful to find the actual value of the error in a result arising from the employment of erroneous data ; or, speculatively, to determine the restrictions to be imposed on probable errors in the data, in order to keep within the limits of error allowable in the result. The expressions derived are for *errors;* hence the sign must be changed to make them *corrections* to be applied to the computed azimuth.

The most important use, however, for these expressions is their application to the investigation of the best and the worst conditions for observation of the celestial body.

28. For convenience, before deducing any equation of maximum and minimum errors in the computed azimuth, we shall derive all the expressions for the resulting error arising from errors in the data—taking each datum separately as variable, the other two given parts as constant—in the several methods of ascertaining the azimuth. It will be convenient to adhere to the familiar common notation of spherical trigonometry—to facilitate the changing of expressions into others equivalent—until the result for each error in the data is reached.

Therefore we shall have with the astronomical triangle a companion triangle, employing the latter to obtain the partial derivatives (see Figs. 1 and 2, art. 18). In the final expressions it is deemed unnecessary to replace dZ, dh, etc., by ΔZ, Δh, etc., to represent finite differences.

29. I. ALTITUDE-AZIMUTH.

Given h, L, *and* d, *to find* Z.

1*st Case.*—Error in Z owing to error in h:

Z and h (B and a) variable;
L and d (c and b) constant.

$$\cos b = \cos c \cos a + \sin c \sin a \cos B \quad . \quad . \quad . \quad . \quad (25)$$

$$0 = -\cos c \sin a\, da + \sin c \cos a \cos B\, da - \sin c \sin a \sin B\, dB;$$

$$\sin c \sin a \sin B\, dB = -(\sin a \cos c - \sin c \cos a \cos B)\, da;$$

$$= -\sin b \cos C\, da;$$

$$dB = -\frac{\sin b \cos C}{\sin c \sin a \sin B}\, da;$$

$$dB = -\frac{\cot C}{\sin a}\, da, \text{ since } \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C};$$

$$dZ = -\frac{\cot q}{\cos h} d\,(90^\circ - h)\,;$$

$$dZ_h = \frac{1}{\tan q \cos h} dh \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

I. 2*d Case.*—Error in Z owing to error in L:

Z and L (B and c) variable;
h and d (a and b) constant.

By interchanging c and a in the preceding case we derive, in the same way as in the 1st case,

$$dZ_L = \frac{1}{\tan t \cos L} dL \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

I. 3*d Case.*—Error in Z owing to error in d:

Z and d (B and b) variable;

h and L (a and c) constant.

From (25), $$-\sin b\, db = -\sin c \sin a \sin B\, dB \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$dB = \frac{\sin b}{\sin c \sin a \sin B} db\,;$$

$$\therefore \text{ since } \frac{\sin b}{\sin B} = \frac{\sin a}{\sin A},$$

$$dB = \frac{1}{\sin c \sin A} db\,;$$

$$dZ = \frac{1}{\cos L \sin t} d\,(90^\circ - d)\,;$$

$$dZ_d = -\frac{1}{\cos L \sin t} dd \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Total error, $dZ = dZ_h + dZ_L + dZ_d$;

$$dZ = \frac{1}{\tan q \cos h} dh + \frac{1}{\tan t \cos L} dL - \frac{1}{\sin t \cos L} dd \quad . \quad . \quad . \quad . \quad . \quad (30)$$

30. II. TIME-AZIMUTH.

Given t, d, *and* L, *to find* Z.

1*st Case.*—Error in Z owing to error in t:

Z and t (B and A) variable;
L and d (c and b) constant.

$$\sin A \cot B = \sin c \cot b - \cos c \cos A. \qquad (31)$$

$$\cos A \cot B\, dA - \sin A \operatorname{cosec}^2 B\, dB = \cos c \sin A\, dA;$$

multiplying by $\sin B$,

$$\cos A \cos B\, dA - \sin A \operatorname{cosec} B\, dB = \cos c \sin A \sin B\, dA;$$

$$dB = \frac{\cos A \cos B - \cos c \sin A \sin B}{\sin A \operatorname{cosec} B} dA;$$

$$dB = -\frac{\cos C}{\sin A \operatorname{cosec} B} dA;$$

$$dB = -\frac{\cos C \sin B}{\sin A} dA = -\frac{\cos C \sin b}{\sin a} dA;$$

$$dZ_t = -\frac{\cos q \cos d}{\cos h} dt. \qquad (32)$$

II. 2*d Case.*—Error in Z owing to error in L:

Z and L (B and c) variable;

t and d (A and b) constant.

From (31), $$-\sin A \operatorname{cosec}^2 B dB = \cos c \cot b\, dc + \sin c \cos A\, dc;$$

$$-\sin A \operatorname{cosec}^2 B dB = \frac{\cos c \cos b + \sin b \sin c \cos A}{\sin b} dc;$$

$$-\frac{\sin A}{\sin^2 B} dB = \frac{\cos a}{\sin b} dc; \qquad (33)$$

from trig., $$\frac{\sin B}{\sin A} = \frac{\sin b}{\sin a}; \qquad (34)$$

$\therefore$ multiplying the first and second members of (33) by the first and second members of (34), respectively,

$$-\frac{1}{\sin B}dB = \cot a\, dc;$$

$$dB = -\cot a \sin B\, dc;$$

$$dZ = -\tan h \sin Z d\,(90 - L);$$

$$dZ_L = \tan h \sin Z dL. \quad \ldots \quad (35)$$

II. 3*d Case.*—Error in Z owing to error in d:

Z and d (B and b) variable;

L and t (c and A) constant.

From (31), $$-\sin A \operatorname{cosec}^2 B\, dB = -\sin c \operatorname{cosec}^2 b\, db;$$

multiplying by $\sin B \sin b$,

$$\frac{\sin A \sin b}{\sin B}dB = \frac{\sin c \sin B}{\sin b}db;$$

by trig., $$\frac{\sin A}{\sin B} = \frac{\sin a}{\sin b} \quad \text{and} \quad \frac{\sin c}{\sin b} = \frac{\sin C}{\sin B};$$

$$\therefore \sin a dB = \sin C\, db;$$

$$\cos h\, dZ = \sin q d\,(90° - d);$$

$$dZ_d = -\frac{\sin q}{\cos h}dd. \quad \ldots \quad (36)$$

Total error, $dZ = dZ_t + dZ_L + dZ_d$.

$$dZ = -\frac{\cos q \cos d}{\cos h}dt + \tan h \sin Z dL - \frac{\sin q}{\cos h}dd. \quad \ldots \quad (37)$$

31. III. TIME-ALTITUDE-AZIMUTH.

Given h, t, *and* d, *to find* Z.

1*st Case.*—Error in Z owing to error in h:

Z and h (B and a) variable;
t and d (A and b) constant.

$$\sin B = \frac{\sin A \sin b}{\sin a}; \quad \ldots \quad (38)$$

$$\cos B\, dB = -\frac{\sin A \sin b \cos a}{\sin^2 a} da;$$

$$\cos B\, dB = -\frac{\sin A}{\sin a} \times \frac{\sin b \cos a}{\sin a} da;$$

$$\cos B\, dB = -\frac{\sin B}{\sin b} \times \sin b \cot a\, da;$$

$$dB = -\tan B \cot a\, da;$$
$$dZ = -\tan Z \tan h\, d(90° - h);$$
$$dZ_h = \tan Z \tan h\, dh. \qquad (39)$$

III. *2d Case.*—Error in Z owing to error in t:

Z and t (A and B) variable;
d and h (b and a) constant.

From (38),
$$\cos B\, dB = \cos A \sin b \operatorname{cosec} a\, dA$$
$$= \frac{\cos A \sin B}{\sin A} dA;$$

$$dB = \cot A \tan B\, dA;$$

$$dZ_t = \cot t \tan Z\, dt. \qquad (40)$$

III. *3d Case.*—Error in Z owing to error in d:

Z and d (B and b) variable;
t and h (A and a) constant.

From (38),
$$\cos B\, dB = \frac{\sin A}{\sin a} \cos b\, db;$$

$$\cos B\, dB = \frac{\sin B}{\sin b} \cos b\, db;$$

$$dB = \tan B \cot b\, db;$$
$$dZ = \tan Z \tan d\, d(90° - d);$$
$$dZ = -\tan Z \tan d\, dd. \qquad (41)$$

Total error, $dZ = \tan Z \tan h\, dh + \tan Z \cot t\, dt - \tan Z \tan d\, dd.$ . . . . . (42)

32. IV. HORIZON-AZIMUTH.

(A special case of altitude-azimuth, in which $h = 0$)

Given h = o, L, *and* d, *to find* Z.

1st Case—Error in Z owing to error in L:

Z and L variable;
h and d constant.

By (27),
$$dZ_L = \frac{1}{\tan t \cos L} dL. \quad (43)$$

For another form, since $h = 0$,
$$\tan t = - \frac{\tan Z}{\sin L};$$

$$\therefore dZ_L = - \tan L \cot Z\, dL. \quad (44)$$

IV. *2d Case.*—Error in Z owing to error in d:

Z and d variable;
h and L constant.

By (29),
$$dZ_d = - \frac{1}{\sin t \cos L} dd. \quad (45)$$

For another form, since $h = 0$,
$$\sin t = \frac{\sin q}{\cos L};$$

$$dZ_d = - \operatorname{cosec} q\, dd. \quad (46)$$

(44) and (46) may also be found from the right spherical triangle PNO, by differentiating $\sin d = \cos Z \cos L$, since the right angle formed by the intersection of the meridian and the horizon is constant, and for another form to (46) we have

$$dZ_d = - \cot Z \cot d\, dd.$$

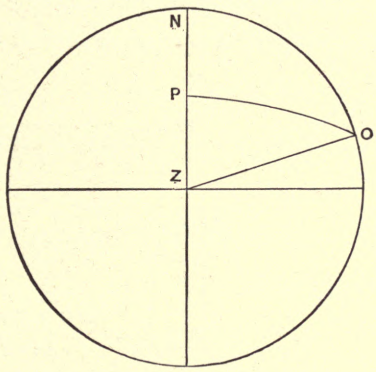

FIG. 3.

$PN = L$;
$NO = Z$;
$PO = p$;
$NOP = 90° - q$;
$PNO = 90°$.

(IV.)

Given L and d, and $PNO = 90°$, to find Z.
1st Case.—Error in Z owing to error in L:

Z and L variable;
$PNO = 90°$, and d constant.

$$\cos Z = \frac{\sin d}{\cos L}; \quad (47)$$

$$- \sin Z\, dZ = \frac{\sin d \sin L}{\cos^2 L} dL;$$

$$- \sin Z\, dZ = \cos Z \frac{\sin L}{\cos L} dL;$$

$$dZ_L = - \tan L \cot Z\, dL \ (44). \quad (48)$$

(IV.) *2d Case.*—Error in Z owing to error in d:

Z and d variable;
$PNO = 90°$, and L constant.

From (47),
$$-\sin Z\, dZ = \frac{\cos d}{\cos L} dd = \frac{\sin Z}{\sin q} dd;$$

$$dZ_d = -\operatorname{cosec} q\, dd \ldots (46). \quad \ldots \ldots \ldots \quad (49)$$

In the triangle PNO,
$$\cos (90° - q) = \frac{\tan Z}{\tan p};$$

$$\therefore \sin q = \frac{\tan Z}{\cot d};$$

$\therefore$ from (49),
$$dZ_d = -\cot d \cot Z\, dd. \quad \ldots \ldots \ldots \quad (50)$$
Total error
$$dZ = -\tan L \cot Z\, dL - \cot d \cot Z\, dd. \quad \ldots \ldots \quad (51)$$

33. The following formulas will sometimes be found useful for purposes of elimination, and reduction of equations that occur subsequently.

By interchanging d and L, Z and q; d and L being constant, we derive, similarly to (26),

$$dq_h = \frac{1}{\tan Z \cos h} dh; \quad \ldots \ldots \ldots \quad (52)$$

and similarly to (32),
$$dq_t = -\frac{\cos Z \cos L}{\cos h} dt. \quad \ldots \ldots \ldots \quad (53)$$

Also from
$$\sin Z = \frac{\sin q \cos a}{\cos L};$$

when L and d are constant,
$$\cos Z\, dZ = \frac{\cos d \cos q}{\cos L} dq;$$

$$dZ_q = \frac{\cos d}{\cos L} \cdot \frac{\cos q}{\cos Z} dq. \quad \ldots \ldots \ldots \quad (54)$$

(54) may also be obtained by eliminating dh from (26) and (52); or by eliminating dt from (32) and (53).

PART IV.

CONSIDERATIONS AFFECTING THE EQUATIONS OF MAXIMUM AND MINIMUM ERRORS, AND RESPECTING THE CURVES OF THESE ERRORS.

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34. Though in *computing the azimuth* it is convenient to regard all the angles as positive within the limit of 180° (see art. 17), on whichever side of the meridian the body is observed, yet looking to the meridian for a possible locus of algebraic maximum and minimum errors occurring—to determine the truth by inspection—it will be necessary to follow the star in its diurnal course and, therefore, to consider the *general astronomical triangle;* necessary, also, in order to discriminate analytically the maximum and minimum errors wherever occurring in the star's path. Following this course, t will be reckoned from 0°, at the upper culmination of the body to the westward to 360°. Z will be reckoned from the point of the horizon nearest the elevated pole to the westward around to 360°, as follows:

If $+d > L$, at the upper transit on the meridian Z passes through 0°, from $Z < 360$ to $Z > 0$; increases to a maximum value $< 90°$, when $q = 90$, then decreases to $\left\{ \begin{matrix} 0° \\ 360° \end{matrix} \right\}$ at lower transit; from 360°, decreases to a minimum value $> 270°$, when $q = 270$, then increases to $\left\{ \begin{matrix} 360° \\ 0° \end{matrix} \right\}$ at upper transit.

If $\pm d < L$, Z at upper transit passes through 180°, from $Z > 180°$ to $Z < 180°$; decreases through 90° to $\left\{ \begin{matrix} 0° \\ 360° \end{matrix} \right\}$ at lower transit; from 360° decreases through 270° to 180° at upper transit.

If $-d > L$, Z at upper transit passes through 180°, from $Z > 180°$ to $Z < 180°$; decreases to a minimum $> 90°$, when $q = 90°$, then increases to 180° at lower transit; increases to a maximum $< 270°$, when $q = 270°$, then decreases to 180° at upper transit.

q will be reckoned to a possible 360° as follows: If $+d > L$, $q = 180°$ at upper transit; decreases through 90° to $\left\{ \begin{matrix} 0° \\ 360° \end{matrix} \right\}$ at lower transit; from 360° decreases through 270° to 180° at upper transit.

If $\pm d < L$, $q = 0$ at upper transit; attains a maximum value $< 90°$, when $Z = 90°$ then decreases to $\left\{ \begin{matrix} 0° \\ 360° \end{matrix} \right\}$ at lower transit; from 360° decreases to a minimum $> 270°$, when $Z = 270°$ then increases to $\left\{ \begin{matrix} 360° \\ 0° \end{matrix} \right\}$ at upper transit.

If $-d > L$, $q = 0$ at upper transit, increases, passing through 90°, to 180° at lower transit; continues to increase through 270° to $\left\{ \begin{matrix} 360° \\ 0° \end{matrix} \right\}$ at upper transit.

Briefly, whenever the given star crosses the meridian of a given place, every one of the

angles t, Z, and q passes through either 0° or 180°; not necessarily the same value for all at the same time. But each angle is $< 180^\circ$ or $> 180^\circ$ according as the other angles are, each, $< 180^\circ$ or $> 180^\circ$; the sides remaining as noted in article 16.

35. For use in *a given observation* the higher equations giving points of maximum and minimum errors will be made to consist of terms in some trigonometric functions of L and d and some *one* function of t (or of h). Substituting the values of L and d that belong to the particular occasion for finding the azimuth, t (or h) can be determined, to know when to observe the body so that the resulting error shall be a minimum; or, to know what point of observation—and its neighborhood—to avoid as giving a maximum error.

For convenience in tracing the locus of the equation, and for the best exhibition of it, L will be regarded as fixed, with d and t (or h) variable: hence, giving different values to d, those of t (or h), corresponding, will be found.

36. In the equation to the locus, with given values of L and d, and calling, for instance, $\sin h = x$, the roots found from the numerical equation embracing both real and imaginary ones, even though every real root correspond to some point of a plane curve regarded as purely algebraic, in which x may have any value, not being restricted to the value of $\sin h$ lying between $+1$ and -1, yet the locus on the sphere may not contain every point represented by the real roots. For if one of these gives $\sin h > \pm 1$, it will present an *impossible* case. Again, the root may have a value not impossible for the trigonometric function, yet one that $\sin h$ never attains. For illustration, when $d = 0$ the greatest value that h can have is $\pm (90^\circ - L)$, when $\sin h = \pm \cos L$; hence, in this case should the root have a numerical value greater than $\cos L$ it will be *inadmissible*.

Therefore, among several real roots there *may* be but one value that is admissible. The problem may also be materially simplified on account of positions of the body that are *impracticable* for observation; for example, the diurnal circle may cross a curve of minimum errors below the horizon; then (theoretically) the best *practicable* position of the body is in the horizon.

37. While equations in the terms mentioned must be resorted to for finding the time or the altitude at which most favorably to observe a given body in the observer's latitude, and though incidentally useful in constructing the curves of maximum and minimum errors, yet, for the latter purpose, equations in terms of Z, L, and h are far simpler.* By them, the point tracing the locus is virtually referred to polar co-ordinates instead of the more complicated bi-polar co-ordinates employed with the equations in terms of t, L, d, and of h, L, d.

But, more than this, the system of polar spherical co-ordinates may readily be transformed into a system of plane rectangular co-ordinates for the *projection of the curve.* By employing the equations to the latter, the difficulties inherent in all the other equations for determining all the properties of the curve are removed, and analysis may be made. Without a knowledge of the properties of the curve of projection, a true conception of the curve on the surface of the sphere is unattainable.

There remain two other forms of equations that are interesting and incidentally useful; namely, (1) the equation referring the point to the system of rectangular spherical co-ordinates, the origin being the zenith, and the spherical axes the meridian and prime vertical; (2) the polar equation to the plane curve of the projection in which θ is the complement of the azimuth and r the linear distance corresponding to the projection of the zenith distance.

38. The diagrams are stereographic projections on the primitive plane of the horizon,

* Insisting on the use of the horizon as the primitive plane. for the best view of the curve (article 38).

the point of sight being at the nadir. The parts lying above the horizon are drawn in full lines; those below the horizon, in broken lines. The distortion in the projection of the lower hemisphere gives an appearance of asymptotic properties to branches of the curves passing through the nadir, and of lack of symmetry in the branches of the loci above and below the horizon. But symmetry exists, and may be more easily perceived if the lower hemisphere is revolved 180° about a tangent at the east point, the point of sight being moved to the new pole of the primitive circle. This can be easily sketched without attempt at great accuracy; but the stereographic projection is considered preferable for the cases presented. In the projection itself the asymptotic properties do exist.

No attempt is made to construct diagrams on a sufficient scale, or with the precision necessary, for use in graphically finding the point to accept as the best, or to avoid as the worst, in observing the body.

39. Tables should be prepared giving the most favorable time for observation, when error in h in the altitude-azimuth and error in t in the time-azimuth are considered, in the case of bodies that cross the prime-vertical in their diurnal course; and tables to give the dividing line between altitude-azimuth and time-azimuth being the better method when error in latitude alone is concerned. Tables applicable to other cases would be useful, but they are not so much needed as are those specified.

40. Since it is purposed to determine, from investigation of the expression for error, the best and the worst conditions for observation—*numerical* maxima and *numerical* minima, irrespective of sign, will often be called *max.* and *min.*; notwithstanding that when found algebraically the *numerical max.* may be an *algebraic min.*, and the *numerical min.* an *algebraic max.*, according to the sign. In general, in this treatise, if the terms *maximum error* and *minimum error* occur unqualified they will mean *numerically greatest* and *least* errors, respectively.

Algebraic max. and min. corresponding to *true* max. and min. where *true* is the term used by mathematicians respecting *max.* and *min.* found by giving to the first differential coefficient the value zero.

Inspection of the given expression will usually show whether it is susceptible of *algebraic* max. and min. If these exist it will sometimes further be seen at exactly what position of the heavenly body they occur. If not obvious, recourse must be had to the analytical method of finding the precise conditions giving them.

But inspection may show that, though no *true* max. or min. exists, yet values of 0 and ∞ do occur, and, for our purpose, these are the *truest* maxima and minima,—sacrificing English to emphasis. To distinguish these from *true* (algebraic) max. and min., and from *finite numerical* max. and min., the term *absolute* will be used in *this treatise*; $dZ = 0$, an absolute min.; $dZ = \infty$, an absolute max.

41. To facilitate comparisons to be made hereafter, it will be convenient to collect the several expressions for errors and to give to each one some equivalent forms from trigonometry not given in what precedes.

I. Altitude-azimuth.

No. 1. $$\frac{dZ}{dh} = \frac{1}{\tan q \cos h} = \frac{\cos q}{\sin q \cos h} = \frac{\cos q}{\sin t \cos L}; \ L \text{ and } d \text{ constant } (26). \quad . \quad . \quad (55)$$

No. 2. $$\frac{dZ}{dL} = \frac{1}{\tan t \cos L} = \frac{\cos t}{\sin t \cos L} = \frac{\cos t}{\sin q \cos h}; \ h \text{ and } d \text{ constant } (27). \quad . \quad . \quad (56)$$

No. 3. $$\frac{dZ}{dd} = -\frac{1}{\sin t \cos L} = -\frac{1}{\sin q \cos h} = -\frac{\cos d}{\sin Z \cos h \cos L}; \ h \text{ and } L \text{ constant } (29). \ (57)$$

II. Time-azimuth.

No. 4. $$\frac{dZ}{dt} = -\frac{\cos q \sin Z}{\sin t} = -\frac{\cos q \cos d}{\cos h} = -\frac{\cos d \sin q \cos q}{\sin t \cos L}; \ d \text{ and } L \text{ constant } (32). \ (58)$$

No. 5. $$\frac{dZ}{dL} = \tan h \sin Z = \frac{\sin h \sin Z}{\cos h} = \frac{\sin h \sin Z \sin q}{\sin t \cos L}$$

$$= \frac{\tan h \sin q \cos d}{\cos L}; \ t \text{ and } d \text{ constant } (35). \quad . \quad (59)$$

No. 6. $$\frac{dZ}{dd} = -\frac{\sin q}{\cos h} = -\frac{\sin t \cos L}{\cos^2 h} = -\frac{\sin^2 q}{\sin t \cos L} = -\frac{\sin q \sin Z}{\sin t \cos d}$$

$$= -\frac{\sin Z \cos L}{\cos h \cos d} = -\frac{\sin^2 Z \cos L}{\sin t \cos^2 d}; \ t \text{ and } L \text{ constant } (36). \quad (60)$$

Inspection of (55), (58), and of (57), (60), shows the truth of the last paragraph of article 12.

III. Time-alt.-azimuth.

No. 7. $$\frac{dZ}{dh} = \tan Z \tan h = \frac{\tan Z \sin q \sin h}{\sin t \cos L} = \frac{\sin q \cos d \sin h}{\cos Z \cos L \cos h}$$

$$= \frac{\sin t \cos d \sin h}{\cos^2 h \cos Z}; \ t \text{ and } d \text{ constant } (39). \quad (61)$$

No. 8. $$\frac{dZ}{dt} = \tan Z \cot t = \frac{\sin Z \cos t}{\cos Z \sin t} = \frac{\cos d \cos t}{\cos Z \cos h}$$

$$= \frac{\cos d \cos t \sin q}{\cos Z \cos L \sin t}; \ h \text{ and } d \text{ constant } (40). \quad (62)$$

No. 9. $$\frac{dZ}{dd} = -\tan Z \tan d = -\frac{\sin q \sin d}{\cos Z \cos L} = -\frac{\sin t \sin d}{\cos Z \cos h}$$

$$= -\frac{\tan Z \sin t \sin d}{\sin Z \cos h} \ (41). \quad (63)$$

Useful expressions when d and L are constant.

$$\frac{dq}{dh} = \frac{1}{\tan Z \cos h} = \frac{\cos Z}{\sin Z \cos h} = \frac{\cos Z}{\sin t \cos d} = \frac{\cos L \cos Z}{\sin q \cos h \cos d} \text{ (52)}. \quad \ldots \ldots \quad (64)$$

$$\frac{dq}{dt} = -\frac{\cos Z \cos L}{\cos h} = -\frac{\cos Z \sin q}{\sin t} = -\frac{\cos Z \sin Z \cos L}{\sin t \cos d} = -\frac{\cos d \sin q}{\cos h \tan Z} \text{ (53)}. \quad \ldots \quad (65)$$

$$\frac{dZ}{dq} = \frac{\cos q \cos d}{\cos Z \cos L} = \frac{\cos q \sin Z}{\cos Z \sin q} = \cot q \tan Z = \frac{\cos h \cos q \tan Z}{\sin t \cos L} \text{ (54)}. \quad \ldots \ldots \quad (66)$$

42. In deriving each of the differential coefficients in what precedes, it was deemed advisable, in the first instance, for the purposes of this work, to use the fundamental equation involving the four parts to be considered and thus find the partial differential required.

But for the multiplicity of cases of differential variations of the astronomical triangle that may arise, the work may be facilitated by employing the three equations (73), (74), and (75), given below, derived as follows:

By spherical trigonometry,

$$\left.\begin{aligned} \cos a &= \cos b \cos c + \sin b \sin c \cos A; \\ \cos b &= \cos a \cos c + \sin a \sin c \cos B; \\ \cos c &= \cos a \cos b + \sin a \sin b \cos C. \end{aligned}\right\} \quad \ldots \ldots \quad (67)$$

Differentiating the first of (67), all parts being variable,

$$\begin{aligned} -\sin a\, da = &-\sin b \cos c\, db - \cos b \sin c\, dc + \cos b \sin c \cos A\, db \\ &+ \sin b \cos c \cos A\, dc - \sin b \sin c \sin A\, dA. \quad . \quad (68) \end{aligned}$$

$$-\sin a\, da = \left\{\begin{aligned} &-(\sin b \cos c - \cos b \sin c \cos A)\, db \\ &-(\sin c \cos b - \cos c \sin b \cos A)\, dc \\ &-\sin b \sin c \sin A\, dA \end{aligned}\right\} = \left\{\begin{aligned} &-\sin a \cos C\, db \\ &-\sin a \cos B\, dc \\ &-\sin a \sin b \sin C\, dA. \end{aligned}\right\} \quad . \quad . \quad (69)$$

Dividing out $\sin a$, $\quad -da = -\cos C\, db - \cos B\, dc - \sin b \sin C\, dA. \quad \ldots \ldots \quad (70)$

In the same way from the second and third of (67), we have

$$-db = -\cos A\, dc - \cos C\, da - \sin c \sin A\, dB. \quad \ldots \ldots \quad (71)$$

$$-dc = -\cos B\, da - \cos A\, db - \sin a \sin B\, dC. \quad \ldots \ldots \quad (72)$$

Whence, substituting angles and complements of the sides in the astronomical triangle,

$$dh = \cos q\, dd + \cos Z\, dL - \cos d \sin q\, dt. \quad \ldots \ldots \quad (73)$$

$$dd = \cos t\, dL + \cos q\, dh - \cos L \sin t\, dZ. \quad \ldots \ldots \quad (74)$$

$$dL = \cos Z\, dh + \cos t\, dd - \cos h \sin Z\, dq. \quad \ldots \ldots \quad (75)$$

Since, in obtaining the partial differential used, we consider two parts of the triangle constant in any given case, it will require not more than two of the equations (70), (71), (72), or of (73), (74), (75), to obtain any one of the differential coefficients.

From (73), (74), (75), some one of the expressions in each group of (55) to (66) will be derived directly.

To obtain other equivalent expressions, the following fundamental formulas are often required:

$$\left.\begin{aligned}
&\sin^2 x + \cos^2 x = 1\,; \\
\cos A &= -\cos B \cos C + \sin B \sin C \cos a\,; \\
\cos B &= -\cos C \cos A + \sin C \sin A \cos b\,; \\
\cos C &= -\cos A \cos B + \sin A \sin B \cos c\,; \\
\cos t &= -\cos Z \cos q + \sin Z \sin q \sin h\,; \\
\cos Z &= -\cos q \cos t + \sin q \text{ sit } t \sin d\,; \\
\cos q &= -\cos t \cos Z + \sin t \sin Z \sin L.
\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (76)$$

PART V.

DETERMINATION, BY INSPECTION, OF THE MOST FAVORABLE AND THE LEAST FAVORABLE POSITIONS OF A GIVEN BODY FOR OBSERVATION IN A GIVEN LATITUDE.

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43. I. ALTITUDE-AZIMUTH.

No. 1. From (55) it is seen that for error in altitude an absolute max. occurs on the meridian for all bodies, $q = 0$ or $180°$: an absolute min., when $q = 90°$ or $270°$; hence, for each of all bodies having $\pm d > L$ at its elongation; but, for these bodies, no algebraic max. or min.

For all bodies whose $d < \pm L$, algebraic max. and min. occur, both being numerical min. Where the min. occurs is not obvious, though text-writers have given overwhelming preponderance to tan q, treating cos h as insignificant, and thus have erroneously assigned the min. error to the position on the prime-vertical. (See articles 5 to 9.)

In the case of $\pm d > L$, when $Z = 90°$ is never attained, incidentally the min. occurs when the body is nearest in azimuth to the prime-vertical. But for $\pm d < L$ recourse must be had to solving $d\left(\frac{1}{\tan q \cos h}\right) = 0$ to obtain the conditions giving algebraic max. and min., which are numerical min. Though the exact point desired (on either side of the meridian) is not obvious, yet it is seen to lie on that side of the prime-vertical towards the nearer pole, since for equal values of q on each side of the prime-vertical $\pm h$ will *there* be less and cos h greater. This branch of the curve of min. errors—from zenith to nadir through the east and west points—together with the branch derived from $q = 90$, giving the entire curve of min. errors, is shown in locus No. 1, the meridian being the locus of max. errors.

It will readily be seen that the text-writers would have *erred to a less degree* had they assigned the most favorable position to the *six-hour circle* instead of to the prime-vertical. For, if $d < L$,

when $Z = 90°$

$$\tan q = \frac{\cot L}{\cos h},$$

and (55) becomes

$$\frac{dZ}{dh} = \tan L. \quad . \quad . \quad . \quad (a)$$

When $t = 90°$

$$\tan q = \frac{\cot L}{\cos d},$$

$$\frac{dZ}{dh} = \frac{\tan L \cos d}{\cos h} = \tan L \sin Z. \quad . \quad . \quad . \quad (b)$$

$(b) < (a)$ excepting for $d = 0°$, when $t = 90°$ and $Z = 90°$ occur at the same time. (See arts. 5 and 15.)

44. The following brief memoranda result from an inspection of the several coefficients of error, L and d being now constant, even though the coefficient is that for error, or change, in one of them; hence that one a variable in getting the expression for error.

No. 2. From (56), $\dfrac{dZ}{dL} = \dfrac{1}{\tan t \cos L}$, for error in L, absolute max. when $t = 0°$ or $180°$; absolute min. when $t = 90°$ or $270°$. No algebraic max. or min. The locus of max. errors, the meridian; that of min. errors, the six-hour circle.

No. 3. From (57), $\dfrac{dZ}{dd} = -\dfrac{1}{\sin t \cos L}$, for error in d, absolute max. when $t = 0°$ or $180°$; numerical min. when $t = 90°$ or $270°$. Loci: the meridian for max.; the six-hour circle for min.

The numerical min. is obvious, but may be verified by finding the algebraic max. and min. The loci being the same as for error in L (No. 2), but the min. error > 0.

45. II. TIME-AZIMUTH.

No. 4. From (58), $\dfrac{dZ}{dt} = \dfrac{-\cos q \cos d}{\cos h}$, for error in t.

For $\pm d > L$, absolute min. when $q = 90°, 270°$.

For $\pm d < L$, numerical min. erroneously assigned to the prime-vertical by some text-writers.

For all bodies algebraic max. and min. generally giving numerical max. when $q = 0, 180°$. Solving $d\left(\dfrac{\cos q \sin Z}{\sin t}\right) = 0$, algebraic max. and min. will be found, giving numerical max. generally when $q = 0, 180°$; and for $d < L$ numerical min., which, though the point is not obvious, proves to lie on that side of the prime-vertical towards the nearer pole. In very high latitudes, for some declinations, this point is very near the meridian and very far from the prime-vertical, in bearing; hence the importance to arctic explorers to know the truth, not to be misled by false teaching.

As in No. 1 (see last paragraph of art. 43) the prime-vertical gives a less favorable position of the body than that on the six-hour circle. For if $d < L$,

when $Z = 90°$,

$$-\frac{dZ}{dt} = \cos q \frac{\cos d}{\cos h}$$
$$= \frac{\tan d}{\tan h} \cdot \frac{\cos d}{\cos h}$$
$$= \frac{\sin d}{\sin h}$$
$$= \sin L. \quad . \quad . \quad . \quad . \quad . \quad (c)$$

When $t = 90°$,

$$-\frac{dZ}{dt} = \cos q \frac{\cos d}{\cos h}$$
$$= \frac{\cot d}{\cot h} \cdot \frac{\cos d}{\cos h}$$
$$= \frac{\cos^2 d}{\cos^2 h} \cdot \frac{\sin h}{\sin d}$$
$$= \frac{\cos^2 d}{\cos^2 h} \sin L$$
$$= \frac{\sin^2 p}{\sin^2 z} \sin L. \quad . \quad . \quad . \quad (d)$$

Excepting when $d = 0$, $t = Z = 90°$, (d) is always less than (c); for when $t = 90°$, z is nearer to $90°$ than is p.

$$\therefore \frac{\sin^2 p}{\sin^2 z} \text{ is a proper fraction.}$$

Loci: Absolute min., curve of $q = 90°$. Numerical min., curve shown in No. 4. Numerical max., the meridian.

No. 5. From (59), $\frac{dZ}{dL} = \tan h \sin Z$, for error in L. Absolute min., when $h = 0°$, also when $Z = 0°$, $180°$. For rising-and-setting bodies numerical max. in each quadrant; for a circumpolar star, a max. on each side of the meridian, above the horizon; for a star always below the horizon, a max. on each side of the meridian. The positions giving numerical max. not obvious, but corresponding to algebraic max. and min. found from $d(\tan h \sin Z) = 0$.

Loci: Absolute min., the horizon and the meridian; numerical max. on curve No. 5.

No. 6. From (60), $\frac{dZ}{dd} = -\frac{\sin q}{\cos h}$, for error in d. Absolute min. when $q = 0°$, $180°$ and h not $90°$. Numerical max. and min. from algebraic max. and min. found from $d\left(\frac{\sin q}{\cos h}\right) = 0$. Inspection alone does not serve to determine even approximately all the positions of the body causing max. and min. errors: but in the single case of $d > L$, it is obvious that there is one max. on each side of the meridian, occurring between the times of the upper culmination of the body and its elongation; since for equal values of $\sin q$ on each side of unity ($q = 90$), $\cos h$ will be less for the greater altitude.

Loci: Absolute min., the meridian; numerical max. and min. given by curve No. 6.

III. Time-altitude-azimuth.

46. *No.* 7. From (61), $\frac{dZ}{dh} = \tan Z \tan h$, for error in h. Absolute max. when $Z = 90°$, $270°$. Absolute min. when $Z = 0°$, $180°$ also when $h = 0$. Numerical max. from algebraic max.

and min. found by $d(\tan Z \tan h) = 0$. There will be a max., then, for any star that does not cross the prime-vertical ($\pm d > L$), on each side of the meridian above and also below the horizon, if a rising-and-setting body. If the body is always visible, a max. above the horizon on each side of the meridian; if always hidden, a max. on each side of the merid. below the horizon. For bodies that cross the prime-vertical above the horizon ($+ d < L$), a max. below the horizon on each side of the meridian; for those that cross below ($- d < L$), a max. above, on each side.

Loci: Absolute max., the prime-vertical; absolute min., the horizon and the meridian; numerical max. by curve No. 7.

No. 8. From (62), $\frac{dZ}{dt} = \frac{\tan Z}{\tan t}$, for error in t. Absolute max., $Z = 90°$ (when t is not $90°$); absolute min., $t = 90°$ (when Z is not $90°$); numerical max. or min. when $Z = 0, 180°$, and $t = 0, 180°$, from algebraic max. and min. found by $d\left(\frac{\cos d \cos t}{\cos Z \cos h}\right) = 0$ or $d\left(\frac{\tan Z}{\tan t}\right) = 0$.

Loci: Absolute max., the prime-vertical; absolute min., the six-hour circle; numerical max. and min., the meridian.

No. 9. From (63), $\frac{dZ}{dd} = - \tan Z \tan d = - \frac{\sin t \sin d}{\cos Z \cos h}$, for error in d. Absolute min., when $Z = 0, 180°$; absolute max., when $Z = 90°, 270°$; numerical max., when $q = 90°$ (limited to bodies having $\pm d > L$), obvious, but may be derived by algebraic max. and min. from $d\left(\frac{\sin t}{\cos Z \cos h}\right) = 0$.

Loci: Absolute min., the meridian; absolute max., the prime-vertical; numerical max., the curve of $q = 90°$.

[NOTE.—In the *Horizon-azimuth*, inspection of (48) and (50) shows that, at a given place, $d = 0$ is the best condition for either error in d or error in L; and that for a given d, the lower the latitude the better.]

47. *The determination of the expressions of maximum and minimum errors in the computed azimuth, due to small errors in the given parts of the astronomical triangle.*

In No. 1, error in h, putting $d\left(\frac{1}{\tan q \cos h}\right) = 0$, it would appear that the second differential is usurping the functions of the first differential coefficient; but it must not be so regarded for this problem of the variation of the triangle. The equation that the first differential is derived from is the general equation giving the relations existing at all times among the parts of the triangle represented, the constants as well as the variables, and it is not an equation conditioned on algebraic max. and min. existing.

The first differential coefficient gives the ratio of change of bearing to change of altitude for a given body in a given latitude. But, this ratio having different values for different positions of the body in its diurnal course, the maximum and minimum values are sought, L and d *remaining* constant. Hence, regarding the first differential as an isolated factor, legitimately derived, which multiplies the error in altitude to produce the error in the computed azimuth, we dispossess it of its character as a differential, regard it as a quantity susceptible

of true max. and min. values, and seek these by putting *its own* first differential equal to zero.

The new variable q thus enters, but its differential is eliminated, knowing the relation existing between it and that of the variable h, while L and d are constants; leaving dh a factor in every term of the derived equation, thus dividing out.

The equation with any assumed latitude is that of a curve, on the surface of the sphere, which is intersected by parallels of declinations within certain limits. The intersections show where the bodies having the declinations corresponding to the parallels must be observed to give algebraic max. or min. errors in the azimuth. To obtain this equation, in whatsoever terms of the astronomical triangle it may be expressed, L and d are assumed constant not only while deriving the partial differential, *but while finding its true max. and min. values;* hence these, in this case, are found by making the second differential coefficient—in the orignal problem of finding simply the general expression for error in altitude—equal to zero.

Not so, however, in the cases of error in L and in d. That we are dealing with *variations* of the astronomical triangle is forcibly shown in these cases. Taking (59) as an instance, $\frac{dZ}{dL} = \tan h \sin Z$, the expression for error in L is the partial differential regarding t and d constants; that is, no error in the declination or in the recorded time of observation. But, having obtained this first differential coefficient, it is not the second differential that is put equal to zero to obtain algebraic max. and min. errors in the azimuth. We do, indeed, put $d(\tan h \sin Z) = 0$; but this must be solved regarding L, *now*, as constant as well as d. For the error in L will have its maximum effect if the star is observed when $\tan h \sin Z$ attains its maximum value, as the star moves with its *unchanged* declination viewed by the observer in his *fixed* latitude—even though his *assumed* latitude is slightly in error. Similar considerations obtain when error in d is concerned.

It is imperative, then, that in putting the differential of the *factor* of the error equal to zero, as, for example, (59) $d(\tan h \sin Z) = 0$ or (60) $d\left(\frac{-\sin q}{\cos h}\right) = 0$, L and d shall be regarded constant in the solution, notwithstanding the factor multiplying the error is derived from a variable L or d; for the change in the aspect of the astronomical triangle is caused by all the parts, other than L and d, varying.

48. The preceding conditions being fulfilled, to determine the relations existing among the several parts of the astronomical triangle to give a max. or min. effect to the coefficient of the error in any given part in the problem,—when any star is observed at any place, the latitude and declination both remaining fixed, while the star travels along its path,—it is evident that the resulting equation expressed in terms of L, d, and any other one part, will give the value of this part required to fulfil the condition of most favorable or least favorable observation.

Hence the most convenient parts to form the equation are L, d, and t, for determining the time to observe. L and d being known, t can be found from the properly selected root of the equation, taking care that t is represented by the same trigonometric function throughout.

The conditions, good for the particular star, hold good for any star whatsoever; hence, giving successive values to d, while L remains constant, the roots of the equation give the hour-angles (t) corresponding; and the curve, on the surface of the sphere, of max. and min. values

of the error in the computed azimuth is the locus of the intersections of the parallels of declination with the hour-circles given by the imposed conditions. Hence, in this sense, we may say that t and d vary together without denying the *unvarying* character of d so far as concerns the establishment of the relations causing the coefficient of error to be a maximum or a minimum. Giving different values to the arbitrary constant L, we shall have on the surface of the sphere a system of curves.

49. For *tracing the curve*, in any particular latitude, the equation to the curve may be turned into terms of L, fixed, and any other two parts varying together. Thus the terms may be in trigonometric functions of L, d, and h, one function adhered to for h, the next best condition to L, d, and t terms for determining at what point to observe in any given case.

But, *for curve-tracing*, the most useful originally derived equation is that consisting of terms in L, h, and Z—the function, for Z being the cosine—for from this equation we can derive the equation to the projection of the curve referred to plane rectangular co-ordinates.

The foregoing remarks apply to all the cases, in the several methods of finding the azimuth, remaining unmentioned. Briefly, the coefficient of error once legitimately derived, its max. or min. effect must be sought by considering that the latitude and declination *do not vary* while the aspect of the astronomical triangle changes with the progress of the star along its path—the diurnal circle = parallel of declination.

50. *Forms of equations to the loci.*

Summary.

(1) In terms of L, d, t; } For use in finding position to observe in any given case.
(2) " " " L, d, h. }
(3) " " " L, h, Z; referred to spherical polar co-ordinates.
(4) derived from (3); referred to spherical rectangular co-ordinates.
(5) derived from (3); referred to plane polar co-ordinates.
(6) " " (3) or (5); referred to plane rectangular co-ordinates.

(1), (2), (3), and (4) all refer to the curve on the surface of the sphere; (5) and (6) to the stereographic projection of the curve.

(4) and (5), though interesting, are of little use in this treatise.

The next step will be to deduce the equations of algebraic max. and min. in the several cases, before discussing the curves they represent for numerical max. or min. The equation to equal numerical effect of error in L, by both time-azimuth and altitude-azimuth, will also be deduced.

PART VI.

EQUATIONS TO THE LOCI OF MAXIMUM AND MINIMUM ERRORS IN THE COMPUTED AZIMUTH DUE TO ERRORS IN THE DATA.

51. *Locus No.* 1. *Altitude-azimuth, error in* h (art. 43). The equation to that part of the locus of numerical mininum error, not obvious, being the locus of algebraie max. and min., when $\pm d < L$.

1st Equation—in terms of L, d, and t. Of the several equivalent forms of (55) any one may be differentiated to give the equation, but care must be taken that a factor essential to give some part of the entire curve shall not be divided out during the operations of simplifying; otherwise, a true branch may be missing in the result: so, also, in the remaining cases (56), (57), etc. The labor of reduction varies with the different forms used. In these pages, the expression giving the least labor, in each case, is retained, after experimenting with the several forms to ascertain which is preferable, as well as to check the result.

Selecting from (55), L being constant, $d\left(\dfrac{\cos q}{\sin t}\right) = 0$;

$$-\sin t \sin q \, dq - \cos q \cos t \, dt = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

Substituting from (65), $\quad dq = -\dfrac{\cos Z \sin q}{\sin t} dt$;

$$\cos Z \sin^2 q - \cos q \cos t = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

$$\left.\begin{aligned} \therefore \cos Z(1 - \cos^2 q) - \cos q \cos t = 0, &\quad \text{turn into } t, L, d; \\ \cos B(1 - \cos^2 C) - \cos C \cos A = 0, &\quad \text{turn into } A, c, b. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (79)$$

By trig., $\quad \cos B = \dfrac{\sin c \cos b - \cos c \sin b \cos A}{\sin a}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$

$$\cos C = \frac{\sin b \cos c - \cos b \sin c \cos A}{\sin a}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

$$\sin^2 a = 1 - \cos^2 a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

Substituting (80) to (83) in (79) and reducing,

$$\left.\begin{aligned}&\cos^4 A + \tan b \cot c\,(\cot^2 b - 2)\cos^3 A - 3\cot^2 c\cos^2 A\\&\qquad + \tan b\cot c\,(2\operatorname{cosec}^2 b + \cot^2 c)\cos A - \operatorname{cosec}^2 b = 0;\\&\cos^4 t + \frac{\tan L}{\tan d}(\tan^2 d - 2)\cos^3 t - 3\tan^2 L\cos^2 t\\&\qquad + \frac{\tan L}{\tan d}(2\sec^2 d + \tan^2 L)\cos t - \sec^2 d = 0.\end{aligned}\right\} \quad (84)$$

52. 2d Equation—in terms of L, d, and h, may be derived from the foregoing, but more conveniently from

$$d\left(\frac{1}{\tan q\cos h}\right) = 0. \quad (85)$$

$$\therefore \tan q\sin h\,dh - \cos h\sec^2 q\,dq = 0. \quad (86)$$

From (64),

$$dq = \frac{1}{\tan Z\cos h}dh. \quad (87)$$

Substituting and dividing out dh,

$$\tan q\sin h - \frac{1}{\tan Z\cos^2 q} = 0. \quad (88)$$

$$\therefore \sin q\cos q\tan Z\sin h - 1 = 0. \quad (89)$$

Substituting

$$\sin q = \frac{\sin Z\cos L}{\cos d},$$

we have

$$\frac{\cos q\sin^2 Z\cos L\sin h}{\cos d\cos Z} - 1 = 0. \quad (90)$$

$$\left.\begin{aligned}&\cos q\,(1 - \cos^2 Z)\cos L\sin h - \cos Z\cos d = 0, \quad \text{turn into } h, L, d;\\&\cos C\,(1 - \cos^2 B)\sin c\cos a - \cos B\sin b = 0, \quad \text{turn into } a, c, b.\end{aligned}\right\} \quad (91)$$

By trig.,

$$\cos B = \frac{\cos b - \cos c\cos a}{\sin a\sin c}. \quad (92)$$

$$\cos C = \frac{\cos c - \cos a\cos b}{\sin a\sin b}. \quad (93)$$

Substituting (92), (93), and (83) in (91), and reducing,

$$\sin^4 h - \frac{\sin L}{\sin d}(\sin^2 d + 2) + 3 \sin^2 L \sin^2 h + \frac{\sin L}{\sin d}(2 \cos^2 d - \sin^2 L) \sin h - \cos^2 d = 0. \quad (94)$$

Note the difference between (84) and (94) in the functions of L and d.

53. 3d Equation—in terms of h, L, and Z.

From (79), $\cos B (1 - \cos^2 C) - \cos C \cos A = 0$, turn into B, a, c.

Multiply by $\sin^2 b$.

$$\sin^2 b \cos B - \sin^2 b \cos B \cos^2 C - \sin b \cos C \,.\, \sin b \cos A. \quad . \quad . \quad . \quad (95)$$

From trig.,

$$\sin b \cos C = \sin a \cos c - \cos a \sin c \cos B. \quad . \quad . \quad . \quad . \quad . \quad . \quad (96)$$

$$\sin b \cos A = \sin c \cos a - \cos c \sin a \cos B. \quad . \quad . \quad . \quad . \quad . \quad . \quad (97)$$

$$\sin^2 b = 1 - \cos^2 b. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98)$$

$$\cos b = \cos c \cos a + \sin c \sin a \cos B. \quad . \quad . \quad . \quad . \quad . \quad . \quad (99)$$

Substituting (96) to (99), as required, in (95) and reducing,

$$\cos^3 B + \cos a \sin a \cot c \cos^2 B - (1 + \sin^2 a \cot^2 c + \cos^2 a) \cos B + \sin a \cos a \cot c = 0. \quad (100)$$

$\therefore$ letting A represent $\tan L$,

$$\cos^3 Z + A \sin h \cos h \cos^2 Z - (1 + A^2 \cos^2 h + \sin^2 h) \cos Z + A \cos h \sin h = 0. \quad . \quad (101)$$

Or, putting zenith-distance in place of the complement of the altitude,

$$\cos^3 Z + A \cos z \sin z \cos^2 Z - (1 + A^2 \sin^2 z + \cos^2 z) \cos Z + A \sin z \cos z = 0. \quad . \quad (102)$$

54. From (102) may readily be derived the equation to the stereographic projection of the curve, referred to plane rectangular co-ordinates.

The primitive plane being the horizon, the rectangular axes X and Y are, respectively, the projections of the prime-vertical and meridian; the origin being the centre of the primitive circle, representing the zenith; the point of sight situated at the nadir. To agree with the accepted reckoning of t and Z to the westward, x should be reckoned positive to the westward and negative to the eastward. Hence, in north latitude x would be positive to the left, negative to the right hand, and the reverse in south latitude; the latter conforming to the conventional reckoning for plane rectangular co-ordinates.

So far as tracing the curve is concerned, since there are always branches symmetrically situated on each side of Y, the meridian, it does not matter which direction for x is arbitrarily chosen as positive; therefore, for convenience, x will be taken positive to the right hand, negative to the left, adhering to the conventional reckoning. y is reckoned positive towards the elevated pole, negative towards the depressed pole.

55. *To obtain the formulas of transformation of equation* (102) *to the equation of the projected curve.*

The radius of the primitive circle being unity, z the zenith distance of any point of the celestial sphere, and r the linear distance, on the projection, of this projected point from the centre of the primitive circle,—we have, from the principles of stereographic projection,

$$r = \tan \tfrac{1}{2}z; \qquad (103)$$

and from plane trig.,

$$\sin z = \frac{2 \tan \frac{1}{2}z}{1 + \tan^2 \frac{1}{2}z}. \qquad (104)$$

$$\cos z = \frac{1 - \tan^2 \frac{1}{2}z}{1 + \tan^2 \frac{1}{2}z}. \qquad (105)$$

$$\tan z = \frac{2 \tan \frac{1}{2}z}{1 - \tan^2 \frac{1}{2}z}. \qquad (106)$$

Since the angle Z on the sphere and in the projection has the same value,

$$\cos Z = \sin \text{co-}Z = \frac{y}{r}; \qquad (107)$$

and, since

$$r^2 = x^2 + y^2, \qquad (108)$$

we have for direct substitution in (101) or (102), to obtain the equation to the projected curve,

$$\sin z = \cos h = \frac{2r}{1 + r^2} = \frac{2\sqrt{x^2 + y^2}}{1 + x^2 + y^2}. \qquad (109)$$

$$\cos z = \sin h = \frac{1 - r^2}{1 + r^2} = \frac{1 - x^2 - y^2}{1 + x^2 + y^2}. \qquad (110)$$

$$\tan z = \cot h = \frac{2r}{1 - r^2} = \frac{2\sqrt{x^2 + y^2}}{1 - x^2 - y^2}. \qquad (111)$$

$$\cot z = \tan h = \frac{1 - r^2}{2r} = \frac{1 - x^2 - y^2}{2\sqrt{x^2 + y^2}}. \qquad (112)$$

$$\cos Z = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2}}; \qquad \sin Z = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}}; \qquad \tan Z = \frac{x}{y}. \qquad (113)$$

$$
\left.\begin{array}{lll}
\text{Let} & \text{A represent } \tan L; & \\
& \text{B} \quad \text{"} \quad \cos L, \text{ whence } & \sqrt{1-\text{B}^2} = \sin L; \\
& \text{C} \quad \text{"} \quad \sin L, \quad \text{"} & \sqrt{1-\text{C}^2} = \cos L.
\end{array}\right\} \quad \ldots\ldots \quad (114)
$$

[NOTE.—A, alone, is required in the cases discussed in this treatise; but based on the principles contained herein, a large number of curves interesting to mathematicians may be found, when considering all the variations of the astronomical triangle, even though the problems lack utility in astronomy. The writer has found, in the case of Time-altitude-latitude azimuth (art. 22), the necessity of employing either B or C, in order to introduce but *one* arbitrary constant (see Appendix).]

It will be found convenient to retain r until its elimination is effected as far as possible before substituting for r^2, r^4, etc., x^2+y^2, $x^4+2x^2y^2+y^4$, etc.

56. To obtain the equation to No. 1, for the projected curve, we have from (101) or (102), substituting, as required, the equations of art. 55,

$$
\frac{y^3}{r^3} + \frac{2\text{A}r(1-r^2)y^2}{r^2(1+r^2)^2} - \left\{1 + \frac{4\text{A}^2r^2}{(1+r^2)^2} + \frac{(1-r^2)^2}{(1+r^2)^2}\right\}\frac{y}{r} + \frac{2\text{A}r(1-r^2)}{(1+r^2)^2} = 0. \quad . \quad (115)
$$

$$
\therefore\ (1+r^2)^2y^3 + 2\text{A}r^2(1-r^2)y^2 - \{r^2(1+r^2)^2 + 4\text{A}^2r^4 + r^2(1-r^2)^2\}y + 2\text{A}r^4(1-r^2) = 0. \quad (116)
$$

$$
\therefore\ y^3 + 2r^2y^3 + r^4y^3 + 2\text{A}r^2y^2 - 2\text{A}r^4y^2 - 2r^2y - 2r^6y - 4\text{A}^2r^4y + 2\text{A}r^4 - 2\text{A}r^6 = 0. \quad (117)
$$

$$
\text{Writing out} \qquad \left.\begin{array}{l}
r^2 = x^2 + y^2; \\
r^4 = (x^2+y^2)^2 = x^4 + 2x^2y^2 + y^4; \\
r^6 = (x^2+y^2)^3 = x^6 + 3x^4y^2 + 3x^2y^4 + y^6.
\end{array}\right\} \quad \ldots\ldots \quad (118)
$$

Substituting, as required, (118) in (117); arranging in order of powers of y, and changing signs throughout, to give positive sign to the first term, we have:

$$
\left.\begin{array}{l}
y^7 + 4\text{A}y^6 + 4x^2y^5 + 4\text{A}^2y^5 - 2y^5 + 10\text{A}x^2y^4 - 4\text{A}y^4 \\
\quad + 5x^4y^3 + 8\text{A}^2x^2y^3 - 2x^2y^3 + y^3 + 8\text{A}x^4y^2 - 6\text{A}x^2y^2 \\
\quad + 2x^6y + 4\text{A}^2x^4y + 2x^2y + 2\text{A}x^6 - 2\text{A}x^4 = 0.
\end{array}\right\} \quad \ldots \quad (119)
$$

Rearranging in order of highest-degree terms,

$$
\left.\begin{array}{l}
y^7 + 4x^2y^5 + 5x^4y^3 + 2x^6y \\
\quad + 4\text{A}y^6 + 10\text{A}x^2y^4 + 8\text{A}x^4y^2 + 2\text{A}x^6 \\
\quad + 4\text{A}^2y^5 - 2y^5 + 8\text{A}^2x^2y^3 - 2x^2y^3 + 4\text{A}^2x^4y \\
\quad - 4\text{A}y^4 - 6\text{A}x^2y^2 - 2\text{A}x^4 \\
\quad + y^3 + 2x^2y = 0.
\end{array}\right\} \quad \ldots\ldots \quad (120)
$$

This is the equation to the locus of algebraic max. and min., giving numerical min., and applies only when $\pm d < L$.

57. For this case, the equation referred to spherical rectangular co-ordinates is of too high a degree to be useful; but in some of the subsequent cases that form will be of interest

if not adding anything to aid in the analysis of the curve. The equation in this form will be introduced in Locus No. 4.

The equation referred to plane-polar co-ordinates may be easily obtained by omitting to substitute for cos Z, in the foregoing work, $\frac{y}{\sqrt{y^2+x^2}}$. But little additional interest will be lent in any case, and for this locus the equation is too complex to be used advantageously.

58. Although not a locus of algebraic max. and min., the curve $q = 90°$ is a branch of the locus of numerical min., giving absolute min. $= 0$, applicable only when $d > L$. This curve, for our purposes, may be called the *curve of elongations*, while known to mathematicians as the *spherical ellipse*.*

The equations are as follows:

$$\cos t = \frac{\tan L}{\tan d}; \quad \sin h = \frac{\sin L}{\sin d}; \quad \cos Z = \frac{\tan L}{\tan h} \quad \ldots \ldots \quad (121)$$

Substituting, from art. 55, in the last of (121), we have for the equation to the projection:

$$y^3 + 2Ay^2 - y + x^2y + 2Ax^2 = 0; \quad \ldots \ldots \ldots \quad (122)$$

or,

$$y^3 + x^2y + 2Ay^2 + 2Ax^2 - y = 0. \quad \ldots \ldots \ldots \quad (123)$$

59. The remaining branch, giving absolute max., the meridian, is clearly defined as the axis Y. No equation is needed; but we have, as a matter of interest,

$$\left.\begin{array}{c} \cos^2 t - 1 = 0, \cos t = \pm 1; \\ \sin h = \begin{cases} \cos (L - d, \\ \quad \text{or} \\ -\cos (L + d; \end{cases} \\ \cos^2 Z - 1 = 0, \ \cos Z = \pm 1. \end{array}\right\} \quad \ldots \ldots \ldots \quad (124)$$

$$\frac{y^2}{x^2+y^2} - 1 = 0, \ y^2 = x^2 + y^2, \ x^2 = 0. \quad \ldots \ldots \ldots \quad (125)$$

For analysis of (120) and (123) see arts. 92, 93.

60. *Locus No.* 2. *Alt.-az., error in* L (art. 44).

There is no branch of algebraic max. and min. For absolute max., the meridian, as in No. 1 (art. 59). For absolute min., the six-hour circle, which is clearly defined by simple construction according to the principles of stereographic projection; yet, as a matter of interest, we have the equations

* For the knowledge that the curve of $q = 90°$ is the spherical ellipse, the writer is indebted to Prof. W. W. Hendrickson, U.S.N.

$$\cos t = 0\,; \qquad \sin h = \sin d \sin L\,; \qquad \cos Z = \frac{\tan h}{\tan L}. \quad . \quad . \quad . \quad . \quad . \quad (126)$$

$$\left.\begin{array}{l} y^2 + 2\mathrm{A}y + x^2 - 1 = 0, \\ y^2 + x^2 + 2\mathrm{A}y - 1 = 0. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (127)$$

or,

Whence, $$y = -\mathrm{A} \pm \sqrt{\mathrm{A}^2 + 1 - x^2}. \quad . \quad . \quad . \quad . \quad . \quad (128)$$

$$x = \pm\sqrt{1 - y^2 - 2\mathrm{A}y}. \quad . \quad . \quad . \quad . \quad . \quad (129)$$

For analysis see art 94.

61. *Locus No.* 3. *Alt.-az., error in* d (art. 44).

The meridian is a branch, giving absolute max. (see art. 59). The six-hour circle is a branch giving algebraic max. and min., always numerical min. This is obvious from (57), $\dfrac{dZ}{dd} = -\dfrac{1}{\sin t \cos L}$; but, to derive the equation, we have $d\left(\dfrac{1}{\sin q \cos h}\right) = 0$;

$$\therefore \sin q \sin h\, dh - \cos h \cos q\, dq = 0. \quad . \quad . \quad . \quad . \quad . \quad (130)$$

By (64), $$dq = \frac{\cos Z}{\sin Z \cos h} dh;$$

$$\sin q \sin h - \frac{\cos q \cos Z}{\sin Z} = 0. \quad . \quad . \quad . \quad . \quad . \quad (131)$$

$$\left.\begin{array}{l} \sin q \sin Z \sin h - \cos q \cos Z = 0\,; \\ \sin C \sin B \cos a - \cos C \cos B = 0. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (132)$$

By trig., the first member of (132) = cos A,

$$\therefore \cos \mathrm{A} = 0\,;$$
$$\cos t = 0, \qquad t = 90° \text{ or } 270°,$$

whence (126) to (129) recur (art. 60).

62. *Locus No.* 4. *Time-azimuth, error in* t (art. 45).

Curve of $q = 90°$ is a branch of absolute min (see art. 58). Curve of algebraic max. and min. giving generally numerical max., the meridian; numerical min. on (?) to be found.

The equation in terms of *t*, *L*, and *d*.

From (58), $$d\left(\frac{\cos q \sin Z}{\sin t}\right) = 0.$$

$$\sin t \sin q \sin Z\, dq - \sin t \cos q \cos Z\, dZ + \cos q \sin Z \cos t\, dt = 0. \quad . \quad . \quad . \quad . \quad (133)$$

From (65) and (58), $dq = -\frac{\cos Z \sin q}{\sin t}dt$; $dZ = -\frac{\cos q \sin Z}{\sin t}dt$;

$$\therefore -\cos Z \sin^2 q \sin Z + \cos^2 q \sin Z \cos Z + \cos q \sin Z \cos t = 0. \quad . \quad . \quad (134)$$

Divide out $\sin Z$ and substitute $1 - \cos^2 q$ for $\sin^2 q$, ... (135)

and we have

$$\left.\begin{aligned} -\cos Z(1 - 2\cos^2 q) + \cos q \cos t &= 0; \\ -\cos B(1 - 2\cos^2 C) + \cos C \cos A &= 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (136)$$

Now turn (136) into t, L, d; A, c, b.

[NOTE.—Though the factor $\sin Z$, which is divided out, gives the meridian when $Z = 0$, yet enough remains to give it still. But, employing $d\left(\frac{\cos q \cos d}{\cos h}\right) = 0$, ($\cos d$ being constant), $\sin Z$ also divides out and leaves nothing to represent the meridian (see art. 51).]

In (136), substituting (80), (81), (82), and (83), and reducing,

$$\left.\begin{aligned} \cos^4 A - 2\tan b \cot c \cos^3 A + \frac{\sin^2 c - \sin^2 b}{\sin^2 b \sin^2 c}\cos^2 A + 2\tan b \cot c \cos A \\ - \frac{\sin^2 c - \sin^2 b \cos^2 c}{\sin^2 b \sin^2 c} = 0; \end{aligned}\right\} (137)$$

$$\left.\begin{aligned} \cos^4 t - 2\cot d \tan L \cos^3 t + \frac{\cos^2 L - \cos^2 d}{\cos^2 L \cos^2 d}\cos^2 t + 2\cot d \tan L \cos t \\ - \frac{\cos^2 L - \cos^2 d \sin^2 L}{\cos^2 d \cos^2 L} = 0. \end{aligned}\right\} (138)$$

Factoring (138),

$$(\cos^2 t - 1)\left(\cos^2 t - 2\tan L \cot d \cos t + \frac{\cos^2 L - \sin^2 L \cos^2 d}{\cos^2 L \cos^2 d}\right) = 0. \quad . \quad . \quad (139)$$

Putting the factor $\cos^2 t - 1 = 0$, $\cos t = \pm 1$; . . . . (140)

therefore the meridian is a branch of the locus of algebraic max. and min., giving, excepting in very high latitudes, always numerical max. (see art. 59).

63. The other factor, solved as a quadratic, gives

$$\cos t = \frac{\tan L}{\tan d} \pm \frac{\sqrt{\sin^2 L - \sin^2 d}}{\sin d \cos d \cos L}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (141)$$

It is obvious that this branch exists only for bodies having $\pm d < L$ $\therefore$ since $\frac{\tan L}{\tan d} > 1$, the positive sign of the radical is inadmissible. This branch is that of numerical minimum.

64. The equation in terms of h, L, and d.

In (139), substitute $$\cos t = \frac{\sin h - \sin L \sin d}{\cos L \cos d} \ldots \ldots \quad (142)$$

The first factor becomes $(\sin h - \sin L \sin d)^2 - \cos^2 L \cos^2 d = 0$; . . . . (143)

$$\therefore \sin h - \sin L \sin d = \pm \cos L \cos d; \ldots \ldots \quad (144)$$

$$\therefore \sin h = \left\{ \begin{array}{l} \sin L \sin d + \cos L \cos d = \cos (L - d), \\ \text{or } \sin L \sin d - \cos L \cos d = - \cos (L + d). \end{array} \right\} \ldots \quad (145)$$

From (145), the meridian is a branch of the locus of algebraic max. and min.

The second factor of (139) becomes, by substituting (142),

$$\sin d \sin^2 h - 2 \sin L \sin h + \sin d = 0; \ldots \ldots \quad (146)$$

$$\therefore \text{ solving as a quadratic, } \sin h = \frac{\sin L \pm \sqrt{\sin^2 L - \sin^2 d}}{\sin d} \ldots \ldots \quad (147)$$

The radical shows that $d > L$ gives an imaginary result; therefore this branch exists only for $d < L$; $\therefore$ since $\frac{\sin L}{\sin d} > 1$, the positive sign of the radical is inadmissible. This branch gives numerical min. The whole curve is given by

$$[(\sin h - \sin L \sin d)^2 - \cos^2 L \cos^2 d] [\sin^2 h \sin d - 2 \sin L \sin h + \sin d]; \quad (148)$$

or
$$\left. \begin{array}{l} \sin^4 h - \dfrac{2 \sin L (1 + \sin^2 d)}{\sin d} \sin^3 h + (5 \sin^2 L + \sin^2 d) \sin^2 h \\ \quad + \dfrac{2 \sin L (1 - \sin^2 L - 2 \sin^2 d)}{\sin d} \sin h - (1 - \sin^2 L - \sin^2 d) = 0. \end{array} \right\} (149)$$

65. The equation in terms of Z, L, h.

Taking (136), $\cos B - 2 \cos B \cos^2 C - \cos C \cos A$, turn into C, B, a.

Multiplying by $\sin^2 b$,

$$\sin^2 b \cos B - 2 \cos B \sin^2 b \cos^2 C - \sin b \cos C \,.\, \sin b \cos A \ldots \quad (150)$$

Substituting (96), (97), (98), (99), in (150) and reducing,

$$\left.\begin{aligned}&\cos B - \frac{\cos a \sin a \cos c}{\sin c(1+\cos^2 a)}\cos^2 B - \cos B + \frac{\cos a \sin a \cos c}{\sin c(1+\cos^2 a)} = 0.\\ &\text{Letting } A = \tan L,\\ &\cos^3 Z - \frac{A \sin h \cos h}{1+\sin^2 h}\cos^2 Z - \cos Z + \frac{A \sin h \cos h}{1+\sin^2 h} = 0.\end{aligned}\right\} \quad . \quad . \quad (151)$$

Factoring, $$(\cos^2 Z - 1)\left(\cos Z - \frac{A \sin h \cos h}{1+\sin^2 h}\right) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (152)$$

The first factor defines the meridian; for the remaining branch we have

$$\cos Z = \frac{A \sin h \cos h}{1+\sin^2 h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (153)$$

66. The equation to the stereographic projection of the branch given by (153). Substituting in (153) as needed, from art. 55, and reducing,

$$y^5 + Ay^4 + 2x^2y^3 + 2Ax^2y^2 - Ay^2 + x^4y + y + Ax^4 - Ax^2 = 0; \quad . \quad . \quad (154)$$

or, rearranging,

$$y^5 + 2x^2y^3 + x^4y + Ay^4 + 2Ax^2y^2 + Ax^4 - Ay^2 - Ax^2 + y = 0. \quad . \quad . \quad . \quad (155)$$

For analysis see art. 95.

67. The equation referred to spherical rectangular co-ordinates.

The axes X and Y being, respectively, the prime-vertical and the meridian, the origin at the zenith, x, in arc, may be reckoned positive to the westward around to 360°, or positive west to 180° and negative east to 180°; y, positive towards the elevated pole up to 90°, the point in the horizon which is the pole to the prime-vertical; negative to 90° towards the other pole.

We have then, $$\cos Z = \sin \text{co-}Z = \frac{\sin y}{\sin z} = \frac{\sin y}{\cos h}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (156)$$

$$\cos z = \sin h = \cos x \cos y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (157)$$

Substituting (156 and (157) in (153),

$$\frac{\sin y}{\cos h} = \frac{A \cos x \cos y \cos h}{\cos^2 x \cos^2 y + 1}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (158)$$

$$\therefore \sin y \cos^2 x \cos^2 y + \sin y = A \cos x \cos y \cos^2 h. \quad . \quad . \quad . \quad . \quad . \quad . \quad (159)$$

Then, since $$\cos^2 h = 1 - \sin^2 h = 1 - \cos^2 x \cos^2 y; \quad . \quad . \quad . \quad . \quad (160)$$

$$\sin y \cos^2 x \cos^2 y + \sin y = \mathrm{A} \cos x \cos y - \mathrm{A} \cos^3 x \cos^3 y; \quad . \quad . \quad . \quad (161)$$

$$\therefore \mathrm{A} \cos^3 y \cos^3 x + \sin y \cos^2 y \cos^2 x - \mathrm{A} \cos y \cos x + \sin y = 0; \quad . \quad . \quad (162)$$

$$\cos^3 x + \frac{\sin y}{\mathrm{A} \cos y} \cos^2 x - \frac{1}{\cos^2 y} \cos x + \frac{\sin y}{\mathrm{A} \cos^3 y} = 0. \quad . \quad . \quad . \quad (163)$$

For the branch $q = 90°$, see art. 58.

68. *Locus No.* 5. *Time-azimuth, error in* L (art. 45).

The meridian and the horizon define the absolute min. For numerical max., the curve of algebraic max. and min. as follows:

The equation in terms of t, L, d.

From (59), $$d(\tan h \sin Z) = 0;$$

$$\therefore \tan h \cos Z dZ + \sin Z \sec^2 h \, dh = 0. \quad . \quad . \quad . \quad . \quad (164)$$

By (55), $$dZ = \frac{1}{\tan q \cos h} dh;$$

$$\left.\begin{aligned} \therefore \sin h + \tan Z \tan q = 0, &\quad \text{turn into } t, L, d; \\ \cos a + \tan B \tan C = 0, &\quad \text{turn into } A, c, b. \end{aligned}\right\} \quad . \quad . \quad . \quad (165)$$

Multiply (165) by $\sin^2 A \cot A \cot B$, and in the resulting second term put $1 - \cos^2 A$ for $\sin^2 A$, and we have

$$\cos a \cot B \sin A \cot C \sin A + 1 - \cos^2 A = 0. \quad . \quad . \quad . \quad (166)$$

By trig., $$\cot B \sin A = \sin c \cot b - \cos c \cos A; \quad . \quad . \quad . \quad . \quad (167)$$

$$\cot C \sin A = \sin b \cot c - \cos b \cos A; \quad . \quad . \quad . \quad . \quad (168)$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A. \quad . \quad . \quad . \quad . \quad (169)$$

In (166), substituting (167), (168), (169), and reducing, we have

$$\left.\begin{aligned} \cos^3 t + \frac{3 \sin^2 L \sin^2 d - \sin^2 L - \sin^2 d - 1}{\sin d \sin L \cos d \cos L} \cos^2 t \\ + (1 - \tan^2 L - \tan^2 d) \cos t + \frac{1 + \sin^2 d \sin^2 L}{\sin d \sin L \cos d \cos L} = 0. \end{aligned}\right\} (170)$$

69. The equation in terms of h, L, d.

From (165), $$\sin h \cos Z \cos q + \sin Z \sin q = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (171)$$

substituting, $$\sin Z = \frac{\sin q \cos d}{\cos L}.$$

$$\left.\begin{array}{l} \sin h \cos Z \cos q \cos L + \sin^2 q \cos d = 0; \text{ turn into } h, L, d; \\ \cos a \cos B \cos C \sin c + \sin b - \sin b \cos^2 C = 0; \text{ turn into } a, c, b. \end{array}\right\} \quad . \quad (172)$$

In (172) substituting (92), (93), and (83) and reducing,

$$\cos^3 a - \frac{1 + \cos^2 c + \cos^2 b}{\cos b \cos c} \cos^2 a + 3 \cos a + \frac{1 - \cos^2 c - \cos^2 b}{\cos b \cos c} = 0; \quad . \quad . \quad (173)$$

$$\sin^3 h - \frac{1 + \sin^2 L + \sin^2 d}{\sin d \sin L} \sin^2 h + 3 \sin h + \frac{1 - \sin^2 L - \sin^2 d}{\sin d \sin L} = 0. \quad . \quad . \quad (174)$$

70. The equation in terms of Z, L, h.
Multiplying (172) by $\sin b$,

$$\cos a \sin c \cos B \sin b \cos C + \sin^2 b - \sin^2 b \cos^2 C = 0. \quad . \quad . \quad . \quad . \quad (175)$$

Substituting (96), (98), (99), and reducing,

$$\cos^2 B - \frac{\cot c \sin a \cos a}{1 + \cos^2 a} \cos B - \frac{1}{1 + \cos^2 a} = 0; \quad . \quad . \quad . \quad . \quad . \quad (176)$$

$$\cos^2 Z - \frac{\tan L \cos h \sin h}{1 + \sin^2 h} \cos Z - \frac{1}{1 + \sin^2 h} = 0. \quad . \quad . \quad . \quad . \quad . \quad (177)$$

71. The equation to the stereographic projection of the curve.
In (177) substitute from art. 55, as needed, reducing and arranging in order of powers of y,

$$\left.\begin{array}{l} y^6 + 2Ay^5 + x^2y^4 - 2y^4 + 4Ax^2y^3 - 2Ay^3 - x^4y^2 \\ \qquad - 4x^2y^2 + y^2 - 2Ax^2y + Ax^4y - x^6 - 2x^4 - x^2 = 0; \end{array}\right\} \quad . \quad (178)$$

or, rearranging in order of terms of highest degree,

$$\left.\begin{array}{l} y^6 + x^2y^4 - x^4y^2 - x^6 + 2Ay^5 + 4Ax^2y^3 + 2Ax^4y \\ \qquad - 2y^4 - 4x^2y^2 - 2x^4 - 2Ay^3 - 2Ax^2y + y^2 - x^2. \end{array}\right\} \quad . \quad (179)$$

For analysis see art. 97.

72. *Locus No.* 6. *Time-azimuth, error in* d (art. 45).

Curve of absolute min., the meridian; curve of numerical max. and min., from algebraic max. and min., as follows:

The equation in terms of t, L, d.

From (60), $$d\left(\frac{\sin q}{\cos h}\right) = 0, \qquad \cos h \cos q\, dq + \sin q \sin h\, dh = 0. \qquad (180)$$

By (64), $$dq = \frac{1}{\tan Z \cos h}\, dh;$$

$$\therefore \frac{\cos q}{\tan Z} + \sin q \sin h = 0; \qquad (181)$$

$$\cos q \cot Z + \sin q \sin h = 0. \qquad (182)$$

Dividing by $\sin q$,

$$\left.\begin{array}{l} \cot q \cot Z + \sin h = 0;\ \text{turn into } t, L, d; \\ \cot C \cot B + \cos a = 0;\ \text{turn into } A, c, b. \end{array}\right\} \qquad (183)$$

Multiply by $$\sin^2 A = 1 - \cos^2 A,$$

$$\sin A \cot C \sin A \cot B + (1 - \cos^2 A) \cos a = 0. \qquad (184)$$

Substituting (167), (168), (169), and reducing,

$$\cos^3 A + (\cot^2 c + \cot^2 b - 1) \cos A - 2 \cot b \cot c = 0; \qquad (185)$$

$$\cos^3 t + (\tan^2 L + \tan^2 d - 1) \cos t - 2 \tan d \tan L = 0. \qquad (186)$$

73. The equation in terms of h, L, d.

Substituting $\cos t = \dfrac{\sin h - \sin d \sin L}{\cos d \cos L}$ in (186),

and we have

$$\sin^3 h - 3 \sin L \sin d \sin^2 h + (2 \sin^2 L + 2 \sin^2 d - 1) \sin h - \sin d \sin L = 0. \qquad (187)$$

74. The equation in terms of Z, L, h.

From (60), $$d\left(\frac{\sin^2 Z}{\sin t}\right) = 0;$$

$$2 \sin t \sin Z \cos Z\, dZ - \sin^2 Z \cos t\, dt = 0. \qquad (188)$$

By (62), $$dZ = -\frac{\cos q \sin Z}{\sin t}\,dt;$$

$$\left.\begin{array}{l} \therefore 2\cos q \cos Z + \cos t = 0;\ \text{turn into } Z, L, h; \\ 2\cos C \cos B + \cos A = 0;\ \text{turn into } B, c, a. \end{array}\right\} \quad \text{(189)}$$

Multiplying by $\sin b$, $$2\sin b \cos C \cos B + \sin b \cos A = 0. \quad \text{(190)}$$

Substituting (96) and (97) in (190) and reducing,

$$\cos^2 B - \tfrac{1}{2}\tan a \cot c \cos B - \tfrac{1}{2} = 0. \quad \text{(191)}$$

$$\cos^2 Z - \tfrac{1}{2}\cot h \tan L \cos Z - \tfrac{1}{2} = 0. \quad \text{(192)}$$

Solving the quadratic, $$\cos Z = \frac{\tan L \pm \sqrt{\tan^2 L + 8\tan^2 h}}{4\tan h}. \quad \text{(193)}$$

75. The equation to the stereographic projection of curve No. 6.

In (192) substituting as needed from art. 55 and reducing,

$$y^4 + 2Ay^3 - y^2 + 2Ax^2y + x^2 - x^4 = 0; \quad \text{(194)}$$

or, $$y^4 - x^4 + 2Ay^3 + 2Ax^2y - y^2 + x^2 = 0. \quad \text{(194a)}$$

For analysis, see art. 98.

76. *Locus No. 7. Time-alt.-azimuth, error in* h (art. 46).

The meridian and the horizon are branches giving absolute min.; the prime-vertical giving absolute max. The branch of numerical max. given by algebraic max. and min. as follows:

From (61), $$d(\tan Z \tan h) = 0;$$

$$\tan Z \sec^2 h\, dh + \tan h \sec^2 Z\, dZ = 0. \quad \text{(195)}$$

By (55), $$dZ = \frac{1}{\tan q \cos h}\,dh,$$

$$\therefore \frac{\sin Z}{\cos Z \cos^2 h} + \frac{\sin h \cos q}{\cos^2 h \cos^2 Z \sin q} = 0. \quad \text{(196)}$$

Substituting, $\sin Z = \dfrac{\sin q \cos d}{\cos L}$,

$$\sin^2 q \cos d \cos Z + \sin h \cos q \cos L = 0. \quad \text{(197)}$$

Since $$\sin^2 q = 1 - \cos^2 q,$$

$$\left.\begin{array}{ll}\cos d \cos Z - \cos d \cos Z \cos^2 q + \sin h \cos q \cos L = 0; & \text{turn into } t, L, d; \\ \sin b \cos B - \sin b \cos B \cos^2 C + \cos a \cos C \sin c = 0; & \text{turn into } A, c, b;\end{array}\right\} . \quad (198)$$

Multiplying by $\sin^3 a$,

$$\sin^2 a \sin b \cos B \sin a - \sin b \sin a \cos B \sin^2 a \cos^2 C + \sin^2 a \cos a \sin c \cos C \sin a = 0. \quad (199)$$

By trig.,
$$\left.\begin{aligned}\cos B \sin a &= \sin c \cos b - \cos c \sin b \cos A; \\ \cos C \sin a &= \sin b \cos c - \cos b \sin c \cos A; \\ \sin^2 a &= 1 - \cos^2 a; \\ \cos a &= \cos b \cos c + \sin b \sin c \cos A.\end{aligned}\right\} \quad \dots\dots \quad (200)$$

Substituting (200) in (199) and reducing,

$$\left.\begin{aligned}&\cos^4 t + \left(4 \tan d \tan L + \frac{\tan^3 L}{\sin d \cos d}\right) \cos^3 t \\ &+ \left\{3 \tan^2 L (\tan^2 d - 1) - \frac{2}{\cos^2 d \cos^2 L}\right\} \cos^2 t \\ &- \left(4 \tan d \tan^3 L + \frac{\tan L \tan d}{\cos^2 d}\right) \cos t \\ &+ \frac{1}{\cos^2 d \cos^4 L} - \tan^2 d \tan^4 L = 0.\end{aligned}\right\} \quad \dots \quad (201)$$

[NOTE.—The writer has done a great amount of "dead work" in attempting to simplify this equation, as well as the other complex equations in this treatise. In the end, of the many various forms obtained in any instance, that form has been retained that seemed, taken all in all, to be the simplest. The remaining forms found are not given here, though perhaps some of them would appear to the reader preferable. The writer would be glad to see the various equations simplified by any one that will attempt the work.

It may be remarked that in performing the operations indicated in this treatise—and believed to be indicated fully as to steps taken—to obtain the equations of high degree in trigonometric functions, there is a vast amount of trigonometric gymnastics required in order to present the equations not more complex than shown. After performing the operations step by step, and collecting the terms (as they then stand) in the order of highest-degree terms of the function of that particular part of the triangle that is given but one function, the coefficients of the different powers of this function are often very complex. Putting the entire computation into these pages would encumber the work.]

In symplifying the coefficients just mentioned, the formula found most useful is $\sin^2 x + \cos^2 x = 1$; and from it $\sin^4 x = \sin^2 x (1 - \cos^2 x)$ and $\cos^4 x = \cos^2 x (1 - \sin^2 x)$ are often needed.

77. The equation in terms of h, L, d.

In (198), substituting (92), (93) and (83), and reducing,

$$\sin^4 h + \frac{\sin^3 L}{\sin d \cos^2 L} \sin^3 h - \frac{2 + \sin^2 L}{\cos^2 L} \sin^2 h + \frac{3 \sin d \sin L}{\cos^2 L} \sin h + \frac{\cos^2 d - \sin^2 L}{\cos^2 L} = 0. \quad (202)$$

78. The equation in terms of Z, L, h.

Multiplying (198) by sin b,

$$\sin^2 b \cos B - \sin^2 b \cos^2 C \cos B + \cos a \sin c \sin b \cos C = 0. \quad . \quad . \quad . \quad . \quad (203)$$

In (203), substituting (96), (98), (99), and reducing,

$$\cos^3 B - \sin^2 a \cos B - \cos a \sin a \cot c = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (204$$

$$\cos^3 Z - \cos^2 h \cos Z - \sin h \cos h \tan L = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (205)$$

79. The equation to the projection.

In (205), substituting from art. 55 as needed, and we have

$$\left.\begin{array}{r} y^7 + 2Ay^6 + 2x^2y^5 - 2y^5 + 6Ax^2y^4 - 2Ay^4 + x^4y^3 - 6x^2y^3 \\ + y^3 + 6Ax^4y^2 - 4Ax^2y^2 - 4x^4y + 2Ax^6 - 2Ax^4 = 0; \end{array}\right\} \quad (206)$$

$$\text{or, } \left.\begin{array}{r} y^7 + 2x^2y^5 + x^4y^3 + 2Ay^6 + 6Ax^2y^4 + 6Ax^4y^2 + 2Ax^6 \\ - 2y^5 - 6x^2y^3 - 4x^4y - 2Ay^4 - 4Ax^2y^2 - 2Ax^4 + y^3 = 0. \end{array}\right\} \quad (207)$$

[For analysis, see art. 99.]

80. *Locus No.* 8. *Time-alt.-azimuth, error in* t (art. 46).

The prime-vertical is a branch for absolute max. The six-hour circle is a branch for absolute min. The branch of numerical max. and min. given by algebraic max. and min., as follows:

The equation in terms of t, L, and d.

From (62), $$d(\tan Z \cot t) = 0;$$

$$\tan t \sec^2 Z \, dZ - \tan Z \sec^2 t \, dt = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (208)$$

By (58), $$dZ = -\frac{\cos q \sin Z}{\sin t} dt;$$

hence, (208) reduces to
$$\left.\begin{array}{r} \cos q \cos t + \cos Z = 0; \\ \cos C \cos A + \cos B = 0; \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (209)$$

which from trig. give
$$\left.\begin{array}{l} \sin A \sin C \cos b = 0; \\ \sin t \sin q \sin d = 0; \text{ turn into } t, L, d. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (210)$$

Substitute $\sin q = \dfrac{\sin t \cos L}{\cos h}$,

and we have
$$\left.\begin{aligned}\sin^2 t \cos L \sin d = 0;\\ (\cos^2 t - 1) \cos L \sin d = 0.\end{aligned}\right\} \quad \ldots \quad (211)$$

Hence with any latitude and declination,

$$\cos^2 t - 1 = 0, \qquad \cos t = \pm 1, \qquad t = 0^\circ,\ 180^\circ;$$

giving the meridian.

81. The equation in terms of h, L, d.

In (211), substitute
$$\cos t = \frac{\sin h - \sin L \sin d}{\cos L \cos d}; \quad \ldots \quad (212)$$

or the same result as follows:

By trig.,
$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}. \quad \ldots \quad (213)$$

In (209), substitute (213), (92), (93), and, reducing, we have

$$\cos b\,(\cos^2 a - 2 \cos c \cos b \cos a + \cos^2 c \cos^2 b - \sin^2 b \sin^2 c) = 0; \quad \ldots \quad (214)$$

$$\left.\begin{aligned}\therefore (\cos a - \cos b \cos c)^2 - \sin^2 b \sin^2 c = 0;\\ (\sin h - \sin d \sin L)^2 - \cos^2 d \cos^2 L = 0;\end{aligned}\right\} \quad \ldots \quad (215)$$

The first factor in (214), sin d, as a constant, divides out; the second factor, put equal to zero and solved, gives, from (215),

$$\sin h - \sin d \sin L = \pm \cos d \cos L; \quad \ldots \quad (216)$$

$$\therefore \sin h = \left\{\begin{aligned}&\sin L \sin d + \cos L \cos d = \cos (L - d) = \cos (d - L),\\ \text{and } &\sin L \sin d - \cos L \cos d = - \cos (L + d),\end{aligned}\right\} \quad . \quad (217)$$

(217) giving the meridian.

82. The equation in terms of Z, L, h.

Multiplying (209) by $\sin^2 b$,

$$\sin b \cos A \sin b \cos C + \sin^2 b \cos B = 0. \quad \ldots \quad (218)$$

Substituting (96), (97), (98) and (99), in (218), and reducing,

$$\left.\begin{aligned}\cos^3 B + \cot c \cot a \cos^2 B - \cos B - \cot c \cot a = 0;\\ \cos^3 Z + \tan L \tan h \cos^2 Z - \cos Z - \tan L \tan h = 0.\end{aligned}\right\} \quad \ldots \quad (219)$$

Factoring (219), $(\cos^2 Z - 1)(\cos Z + \tan L \tan h) = 0.$ . . . . (220)

The first factor gives the meridian, $\cos Z = \pm 1$, $Z = 0$, $180°$. The second factor gives the equator from

$$\left.\begin{aligned} \cos Z \cos L \cos h + \sin L \sin h &= 0; \\ \cos B \sin c \sin a + \cos c \cos a = 0 &= \cos b \text{ (by trig.)}; \end{aligned}\right\} \quad \ldots \quad (221)$$

$\therefore \sin d = 0$, giving equator. But, as will be shown, this is not a branch giving max. and min., but giving a constant value to dZ; and the equator divides the meridian into parts such that the true max. and min. at transits of stars are for $+d$ numerical max. at upper culmination, numerical min. at lower culmination, and *vice versa* for $-d$, provided we do not yet consider the intervening absolute max. and min.

83. Although the equator is clearly defined and easily constructed in the projection, yet we may find the equation to its projection. From (220), $\cos Z + \tan L \tan h = 0$, and by art. 55,

$$\frac{y}{r} + \frac{A(1 - r^2)}{2r} = 0; \quad \ldots \quad (222)$$

$$2y + A - Ax^2 - Ay^2 = 0;$$

$$\therefore \quad y^2 + x^2 - \frac{2}{A}y - 1 = 0. \quad \ldots \quad (223)$$

A star whose $d = 0$ is as favorably situated at one point as at any other in its diurnal course. This may be shown directly from (62),

$$\frac{dZ}{dt} = \frac{\cos t \cos d}{\cos Z \cos h}. \quad \ldots \quad (224)$$

By trig.,

$$\left.\begin{aligned} \sin a \cos B &= \sin c \cos b - \cos c \sin b \cos A; \\ \therefore \cos h \cos Z &= \cos L \sin d - \sin L \cos d \cos t; \end{aligned}\right\} \quad \ldots \quad (225)$$

when $d = 0$, $\cos h \cos Z = -\sin L \cos t.$ . . . . (226)

Substituting (226) in (224), $\cos d$ being unity,

$$\frac{dZ}{dt} = -\frac{1}{\sin L}; \quad \ldots \quad (227)$$

a constant quantity in a given latitude. For analysis, see art. 100.

84. *Locus No.* 9. *Time-alt.-azimuth, error in* d (art. 46).

The meridian is a branch of absolute min. The prime-vertical is a branch of absolute max. The branch of algebraic max. and min. exists only for $\pm d > L$, and it is obviously

the curve $q = 90°$, as shown in (63), $\frac{dZ}{dd} = -\tan Z \tan d$. Though obvious, we may find this branch analytically, as one of algebraic max. and min., as follows: choosing from (63), we have

$$d\left(\frac{\sin q}{\cos Z}\right) = 0;$$

$$\therefore \cos Z \cos q\, dq + \sin q \sin Z\, dZ = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (228)$$

By (66),

$$dZ = \frac{\cos q \cos d}{\cos Z \cos L} dq;$$

$$\therefore \cos Z \cos q + \frac{\sin q \sin Z \cos q \cos d}{\cos Z \cos L} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (229)$$

Substitute

$$\sin q = \frac{\sin Z \cos L}{\cos d}$$

and clear of fractions, when we have

$$\left.\begin{array}{l} \cos^2 Z \cos q + \sin^2 Z \cos q = 0; \\ \left(\underbrace{\cos^2 Z + \sin^2 Z}_{=1}\right) \cos q = 0. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (230)$$

$$\left.\begin{array}{l} \therefore \cos q = 0, \qquad q = \left\{\begin{array}{l} 90° \\ 270° \end{array}\right. \\ \cos t = \frac{\tan L}{\tan d}; \\ \sin h = \frac{\sin L}{\sin d}; \\ \cos Z = \frac{\tan L}{\tan h}; \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (231)$$

For the equation to the projection of $q = 90$, see art. 58, Locus No. 1, where this curve exists for an absolue min. Eq. (123), $y^3 + x^2y + 2Ay^2 + 2Ax^2 - y = 0$ (see art. 101).

85. *Locus No.* 10. *Time-azimuth and altitude-azimuth giving equal numerical values to the error in the computed azimuth, arising from a small error in the latitude; whence the limits within which the one method is more favorable than the other, so far as the error in latitude is concerned.*

By inspection of (56) and (59), taking for investigation the single case of a rising-and-setting body having $+d > L$, we see that in the method of altitude-azimuth the error when the body is in the horizon at rising, t lying between 180° and 270°, has a *positive finite* value, decreasing to zero when t becomes equal to 90° as the star travels in its diurnal path: by the

time-azimuth, when the star is in the horizon the value of the error is zero, and then it increases negatively to a *negative finite* value when $t = 90°$; Z throughout this time remaining between 360° and 270°.

Therefore, (1) for some point of observation (e') in the star's path, between the horizon and the six-hour circle, the values of the error in the azimuths, computed with both methods separately, will be numerically equal, with contrary signs; (2) it is obvious also that before the star reaches the meridian, after passing (e'), some point of observation (f') will give identical values to the errors, that is, numerically equal, having the same sign.

The time-azimuth then will be preferable from the time of rising of the star until e' is reached; thenceforward, passing through $t = 90°$ and up to (f'), the altitude-azimuth the better method; thereafter to the meridian we should employ the time-azimuth. We may, in a similar way, follow the star west of the meridian.

Our object, now, will be to find the equations to the curves of equal errors with opposite signs, and with like signs; whence, constructing the curves, the limits within which either method is preferable will be graphically given.

86 (a). *First.—Identical errors—signs alike.*

Putting (56) and (59) equal to each other,

$$\frac{1}{\tan t \cos L} = \tan h \sin Z; \quad \ldots \quad (232)$$

$$\left.\begin{aligned} \cot t - \tan h \sin Z \cos L &= 0; \\ \cot A - \cot a \sin B \sin c &= 0. \end{aligned}\right\} \quad \ldots \quad (233)$$

Multiplying by $\sin B$, $$\sin B \cot A - \cot a \sin^2 B \sin c = 0. \quad \ldots \quad (234)$$

By trig., $$\left.\begin{aligned} \sin B \cot A &= \sin c \cot a - \cos c \cos B; \\ \sin^2 B &= 1 - \cos^2 B. \end{aligned}\right\} \quad \ldots \quad (235)$$

Substituting (235) in (234) and reducing,

$$\left.\begin{aligned} \cos^2 B \cot a \sin c - \cos B \cos c &= 0; \\ \cos^2 Z \tan h \cos L - \cos Z \sin L &= 0; \\ \cos Z(\cos Z \tan h \cos L - \sin L &= 0. \end{aligned}\right\} \quad \ldots \quad (236)$$

The first factor of (236) gives $$\cos Z = 0; \quad \ldots \quad (237)$$

$\therefore Z = 90°, 270°$, and the prime-vertical is a branch of the locus.

Putting the second factor equal to zero, we have

$$\cos Z = \frac{\tan L}{\tan h}, \quad \ldots \quad (238)$$

which value can exist only when $q = 90°$, $270°$; hence the curve of elongations, $q = 90°$, is a branch of the locus.

Hence the prime vertical is a branch for all bodies having $\pm d < L$, and we have for conditions,

$$\cos t = \frac{\tan d}{\tan L}, \qquad \sin h = \frac{\sin d}{\sin L}; \quad \ldots \ldots \ldots \ldots \quad (239)$$

and the curves of elongations are branches for $\pm d > L$, giving

$$\cos t = \frac{\tan L}{\tan d}, \qquad \sin h = \frac{\sin L}{\sin d}. \quad \ldots \ldots \ldots \ldots \quad (240)$$

From (238) we have, as in (123), the equation to the projection of the curve of elongations,

$$y^3 + x^2y + 2Ay^2 + 2Ax^2 - y = 0;$$

and from (237) for the prime-vertical $y = 0$ equation to the projection.

86 (b). The branches found in art. 86 (a) may be determined with less labor by selecting from (56) and (59) as follows:

$$\frac{\cos t}{\sin t \cos L} = \frac{\sin h \sin Z \sin q}{\sin t \cos L} \quad \ldots \ldots \ldots \ldots \quad (232)b$$

but both methods are retained as checks upon each other.

From (232)b we have

$$\left. \begin{aligned} \cos t - \sin h \sin Z \sin q &= 0; \\ \cos A - \cos a \sin B \sin C &= 0. \end{aligned} \right\} \quad \ldots \ldots \ldots \ldots \quad (233)b$$

By trig.,

$$-\cos B \cos C = (233)b;$$

$$\left. \begin{aligned} \therefore \cos B \cos C &= 0; \\ \cos Z \cos q &= 0. \end{aligned} \right\} \quad \ldots \ldots \ldots \ldots \quad (236)b$$

$\therefore \cos Z = 0$ (gives the prime-vertical, and)

$\cos q = 0$ (gives the curve of elongations.)

87. *Second.—Equal errors, numerical, opposite signs.*

Changing the sign of one member in (232)b, and cancelling $\sin t \cos L$, we have

$$\left. \begin{aligned} \cos t &= -\sin h \sin Z \sin q; \\ \cos A + \cos a \sin B \sin C &= 0. \end{aligned} \right\} \quad \ldots \ldots \ldots \ldots \quad (241)$$

For the equation in terms of t, L, d; A, c, b.

By trig., $\sin B = \dfrac{\sin A \sin b}{\sin a}$ and $\sin C = \dfrac{\sin A \sin c}{\sin a}$;

substituting in (241), $\cos A + \dfrac{\cos a \sin b \sin c \sin^2 A}{\sin^2 a} = 0.$

Substituting $\sin^2 a = 1 - \cos^2 a$, and $\sin^2 A = 1 - \cos^2 A$,

and clearing, we have

$$\cos A - \cos^2 a \cos A + \sin b \sin c \cos a - \sin b \sin c \cos a \cos^2 A = 0. \quad . \quad . \quad (242)$$

Since $\cos a = \cos b \cos c + \sin b \sin c \cos A$,

we have
$$\left.\begin{aligned} &\cos A - \cos^2 b \cos^2 c \cos A - 2 \cos b \cos c \sin b \sin c \cos^2 A \\ &\quad - \sin^2 b \sin^2 c \cos^3 A + \sin b \sin c \cos b \cos c + \sin^2 b \sin^2 c \cos A \\ &\quad - \sin b \sin c \cos b \cos c \cos^2 A - \sin^2 b \sin^2 c \cos^3 A = 0. \end{aligned}\right\} \quad (243)$$

Arranging in order of powers of $\cos A$, collecting terms,

$$\left.\begin{aligned} &- 2 \sin^2 b \sin^2 c \cos^3 A - 3 \sin b \sin c \cos b \cos c \cos^2 A \\ &+ (1 - \cos^2 b \cos^2 c + \sin^2 b \sin^2 c) \cos A \,.\, + \sin b \sin c \cos b \cos c = 0. \end{aligned}\right\} \quad . \quad (244)$$

The coefficient of $\cos A$ reduces to $\sin^2 b + \sin^2 c$, and dividing through by $- 2 \sin^2 b \sin^2 c$

we have

$$\left.\begin{aligned} &\cos^3 A + \tfrac{3}{2} \cot b \cot c \cos^2 A - \tfrac{1}{2} (\operatorname{cosec}^2 c + \operatorname{cosec}^2 b) \cos A - \tfrac{1}{2} \cot b \cot c = 0; \\ &\cos^3 t + \tfrac{3}{2} \tan d \tan L \cos^2 t - \tfrac{1}{2} (\sec^2 L + \sec^2 d) \cos t - \tfrac{1}{2} \tan d \tan L \quad = 0. \end{aligned}\right\} \quad (245)$$

88. The equation in terms of h, L, d; a, b, c.

In (242), substituting $\cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}$,

performing the operations, collecting terms, and simplifying, we have

$$\left.\begin{aligned} &\cos^3 a - \tfrac{3}{2} \cos b \cos c \cos^2 a - \frac{\sin^2 c + \sin^2 b}{2} \cos a + \tfrac{1}{2} \cos b \cos c = 0; \\ &\sin^3 h - \tfrac{3}{2} \sin d \sin L \sin^2 h - \tfrac{1}{2} (\cos^2 L + \cos^2 d) \sin h + \tfrac{1}{2} \sin L \sin d = 0. \end{aligned}\right\} \quad (246)$$

89. The equation in terms of Z, L, h; B, c, a.
Changing the sign of the second term of (233),

$$\left.\begin{array}{l}\cot t + \tan h \sin Z \cos L = 0; \\ \cot A + \cot a \sin B \sin c = 0.\end{array}\right\} \quad \ldots\ldots \quad (247)$$

Multiplying by sin B, $\sin B \cot A + \cot a \sin^2 B \sin c = 0. \quad \ldots\ldots \quad (248)$

Substituting (235) in (248), and reducing,

$$\cos^2 B + \cot c \tan a \cos B - 2 = 0; \quad \ldots\ldots \quad (249)$$

$$\cos^2 Z + \tan L \cot h \cos Z - 2 = 0. \quad \ldots\ldots \quad (250)$$

90. The equation to the projection of the curve of equal numerical errors, contrary signs. Substituting in (250) from art. 55, as required, and reducing, we have

$$\frac{y^2}{r^2} + \frac{2A}{1 - r^2} y - 2 = 0. \quad \ldots\ldots \quad (251)$$

$$y^4 + 2Ay^3 + 3x^2y^2 - y^2 + 2Ax^2y - 2x^2 + 2x^4 = 0; \quad \ldots\ldots \quad (252)$$

or, rearranging, $y^4 + 3x^2y^2 + 2x^4 + 2Ay^3 + 2Ax^2y - y^2 - 2x^2 = 0 \quad \ldots\ldots \quad (253)$

For analysis see art. 102.

PART VII.

ANALYSIS OF THE EQUATIONS TO THE LOCI, AND THE TRACING OF THE CURVES.

91. To Lieutenant H. O. Rittenhouse, U.S.N., the writer is greatly indebted for assistance, both mathematical and constructive, in analyzing the equations and tracing the loci. In the progress of delineating the latter, Lieutenant Rittenhouse gave much aid in removing difficulties encountered. The drawings for the accompanying plates were made by him, and they represent the curves faithfully for the particular latitudes given, to serve as illustrations.

For the determination of the most favorable and the least favorable conditions for observation of the star, by finding the algebraic maximum and minimum effects of the errors in the data, the writer claims originality. To the fact that text-writers have been content to ascertain (?) by inspection of the first differential of the particular equation employed in the problems for time, latitude, and azimuth what conditions are the best,—is due the fallacy of accrediting the prime-vertical the place of best observation, when error in h and error in t are concerned, in the altitude-azimuth and the time-azimuth.

Having undertaken this work, the problem of finding, in any given case, the best position for observation demanded the equation in t, L, d, or, as an alternative, in h, L, d, for L and d must be known and used. By means of these two equations considerable progress was made in describing the curves: where one equation failed to clear some doubtful point, the other would sometimes come to the rescue. But, still, there always remained some points in obscurity; inciting guesses where reason failed: some guesses being very good, as ultimately proved; others as bad as could be. All the curves had been defined (?) by these equations, and the writer was continuing a great amount of " dead work" among them, in the hope that some one curve, at least, simpler than its fellows, would prove accommodating—when it occurred to him that the tracing of the curve need have no dependence on the problem of finding, in a given case, the time to observe at, or the altitude to observe; but that, in a given latitude, by varying the altitudes for *points of the curve*, the corresponding azimuths might be found and therefore the curve could be traced. Consequently, the equations were derived in terms of Z, L, and h; and it was then discovered that these equations could be easily transformed into the equations in terms of x, y, and one arbitrary constant, tan L, for referring the projection of the curves to plane rectangular co-ordinates.* If this were not accomplished, still the equations in Z, L, h, were much more useful than the others, for they re-

* See foot-note to art. 37.

ferred to the more simple system of spherical polar co-ordinates; and, at the same time, in general, they dropped one degree lower than those in t, L, d and h, L, d.

The equations to the projection, once obtained, the tracing of the curves became comparatively easy, notwithstanding some of the equations are of high degree and involve some nice points. It is not believed that all these points have been discovered, and the writer will be gratified to see newly discovered truths respecting the curves presented. Considered as plane curves, simply,—losing sight of the problem from which they are evolved,—they possess very interesting features. Based on the same principles giving *these* curves, a great number of others may be discovered by considering all manner of variations in the astronomical triangle—outside of the utility problems. (For this subject, see appendix.)

Each curve is given for latitudes 30°, 45°, 60°, thus fairly showing the alterations of form between these stages as the curve gradually changes between the limiting latitudes of 0° and 90°. In the *growth* of the curves few unlooked-for changes of form occur. In No. 1 and No. 4 some surprises are met with. In No. 4, apart from analysis, the change in form between 30° and 60° of latitude graphically hints at a great change in higher latitudes.

92. *Locus No.* 1. *Alt.-azimuth, error in* h. Articles 43 and 51 to 56. Branches—1. Algebraic max. and min., giving numerical min. 2. Absolute min., curve of elongations. 3. Absolute max., the meridian.

1st Branch.

(*a*) By (120),

$$y^7 + 4x^2y^5 + 5x^4y^3 + 2x^6y$$
$$+ 4Ay^6 + 10Ax^2y^4 + 8Ax^4y^2 + 2Ax^6$$
$$+ 4A^2y^5 - 2y^6 + 8Ax^2y^3 - 2x^2y^3 + 4Ax^4y$$
$$- 4Ay^4 - 6Ax^2y^2 - 2Ax^4$$
$$+ y^3 + 2x^2y = 0. \qquad (254)$$

From terms of highest degree, $y(y^2 + x^2)^2 (y^2 + 2x^2) = 0. \qquad (255)$

Hence, there exists an infinite branch having an asymptote parallel to X; and no other real infinite branch.

Coefficient of highest power of x is $2y + 2A$;

$$\therefore y = - A \text{ for the asymptote}, \qquad (256)$$

which is the projection of the parallel of declination, $- d = L. \qquad (257)$

For $-y = \tan L$, and disregarding sign of y, let $z =$ arc from the origin to the point a where the asymptote cuts the meridian, Y ; then

$$\tan \tfrac{1}{2} z = y = \tan L, \therefore z = 2L. \quad \ldots \ldots \quad (258)$$

$\therefore z = L + L$. On Y the arc from the origin to the equator $= L$; from equator to the point a, the arc = dec. of point a = the remaining L, $\therefore$ resuming sign, $-d = L$.

(*b*) For points of locus on Y, put $x = 0$, $\therefore$

$$y^7 + 4Ay^6 + 4A^2y^5 - 2y^5 + 4Ay^4 + y^3 = 0; \quad \ldots \ldots \quad (259)$$

$$\therefore y^3 (y^2 + 2Ay - 1)^2 = 0. \quad \ldots \ldots \quad (260)$$

First factor gives $y^3 = 0$, the origin.

Second factor gives $$y = -A \pm \sqrt{A^2 + 1}. \quad \ldots \ldots \quad (261)$$

N. and S. poles, P and P′. (Imaginary, see (*f*)).

(*c*) For points on X, put $y = 0$.

$$2Ax^6 - 2Ax^4 = 0;$$

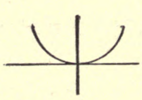
$$\therefore x^4 = 0, \text{ the origin}; \; x^2 = \pm 1, \text{ E. and W. points.} \quad \ldots \ldots \quad (262)$$

(*d*) For form at origin, from terms of lowest degree:

$$y^3 + 2x^2y = 0, \qquad \text{or} \qquad y(y^2 + 2x^2) = 0, \text{ (imaginary).} \quad \ldots \quad (263)$$

But from $$2yx^2 - 2Ax^4 = 0. \quad \ldots \ldots \quad (264)$$

$$y = Ax^2. \quad \ldots \ldots \quad (265)$$

And the curve is of the form  (Fig. 4).

(*e*) Form of curve at E. point.

Moving origin to (1, 0) the coefficient of x is found to be $4A$; the coefficient of y, $4A^2 + 4$,

$$\therefore y = -\frac{A}{1 + A^2} x \text{ is the tangent at E.} \quad \ldots \ldots \quad (266)$$

Similarly, $$y = \frac{A}{1 + A^2} \text{ is the tangent at W.} \quad \ldots \ldots \quad (267)$$

(f) To ascertain the character of the locus at P and P′; moving origin to P, putting for $y, y - A + \sqrt{A^2+1}$, the coefficient of y vanishes. We have for determining the sign of the coefficient of y^2,

$$\{2A^4 + 3A^2 + 1 - 2A^3\sqrt{A^2+1} - 2A\sqrt{A^2+1};\}$$

similarly for x^2, $\{2A^4 + 3A^2 + 1 - 2A^3\sqrt{A^2+1} - 2A\sqrt{A^2+1}.\}$

The signs of these two terms, therefore, will be always the same, and the form at P is imaginary. P is therefore a peculiar point, and, from the symmetry of the locus, P′ must also be a peculiar point.

(g) Though no real branch of this curve passes through P or P′, curves to follow do pass through these points; therefore, to prove that the N. and S. poles are given by $y = -A \pm \sqrt{A^2+1}$, we have for P, the N. pole, co-L = arc from the origin (zenith) to the pole. By stereographic projection, if r is the linear distance from origin to P,

then $$r = \tan \tfrac{1}{2}\,\text{co.-}L. \qquad (268)$$

By trig., $$\tan \tfrac{1}{2}\,\text{co-}L = \frac{1 - \cos \text{co-}L}{\sin \text{co-}L} = \frac{1 - \sin L}{\cos L}, \qquad (269)$$

also $$-A + \sqrt{A^2+1} = -\frac{\sin L}{\cos L} + \sqrt{\frac{\sin^2 L + \cos^2 L}{\cos^2 L}} \qquad (270)$$

$$= \frac{1 - \sin L}{\cos L}; \quad \therefore r = y. \qquad (271)$$

Similarly for P′,

$$y = -A - \sqrt{A^2+1}, \quad \text{or} \quad -y = A + \sqrt{A^2+1} = \frac{1 + \sin L}{\cos L}; \qquad (272)$$

r' = distance from origin to P′ = $\tan \frac{1}{2}(90° + L)$.

By trig., $$\tan \tfrac{1}{2}(90° + L) = \frac{1 - \cos(90° + L)}{\sin(90° + L)} = \frac{1 + \sin L}{\cos L}; \qquad (273)$$

$$r' = -y. \qquad (274)$$

(h) The curve does not cross its asymptote, for, combining (256) with (254), we have

$$(A^3 + 2A)x^4 + 2A(A^2+1)^2x^2 + A^3(A^2+1)^2 = 0, \qquad (275)$$

which gives imaginary roots only.

(i) For limiting form of curve, $L = 0$, put $A = 0$ in (254),

$$y = 0 \text{ divides out}; \qquad (276)$$

$\therefore$ axis of X, the prime-vertical, is a part of the locus. There remains

$$y^6 + 4x^2y^4 + 5x^4y^2 + 2x^6 - 2y^4 - 2x^2y^2 + y^2 + 2x^2 = 0. \qquad (277)$$

Infinite branches are imaginary and form at origin imaginary.

If $\quad y = 0, \quad x^2 = 0, \qquad (278)$

and $\quad x = \sqrt[4]{-1}$, *imaginary*. $\qquad (279)$

If $\quad x = 0, \quad y^2 = 0, \qquad (280)$

and $\quad (y^2 - 1)^2 = 0, \quad y = \pm 1$, double point at N. and S. $\qquad (281)$

Moving origin to N., putting $y = y + 1$, the form is given by

$$x^2 + y^2 = 0; \qquad (282)$$

$\therefore$ *imaginary* branch at N. point, the same at S. point; N. and S. are isolated points.

Arranging (277) as a cubic in x^2, we have

$$2x^6 + 5y^2x^4 + (4y^4 - 2y^2 + 2)x^2 + y^2(y^4 - 2y^2 + 1) = 0. \qquad (283)$$

These coefficients are all positive, therefore x^2 can have no positive root, and therefore x can have no real root. The locus is therefore imaginary, excepting the line $y = 0$ (that is X, the prime-vertical which is the equator), and the peculiar points at zenith and N. and S.

(k) For limiting form $L = 90°$. Put $A = \infty$ in (254) and we have

$$y(y^4 + 2x^2y^2 + x^4), \quad \text{or} \quad y(x^2 + y^2)^2 = 0; \qquad (284)$$

$$\therefore y = 0, \text{ giving X, the prime-vertical}; \qquad (285)$$

$$(x^2 + y^2)^2 = 0, \text{ giving origin, zenith, a double conjugate point}. \qquad (286)$$

(l) Not considering detached points, it is seen from (i) and (k) that starting with Lat. $= 0$, the axis of X is the locus; and ending with Lat. $= 90°$, again the axis of X is the locus; while between these limiting latitudes there is always one branch passing through the origin

and E. and W., and always having an asymptote $y = -A$. On the sphere it is a continuous branch through zenith, E., nadir, W., to zenith.

The question arises: How does the curve change its shape with the change of latitude, *so that it shall return to its original form?*

[NOTE.—For the following elucidation and the determination of the envelope, the writer is much indebted to Lieutenant Rittenhouse and Professor Hendrickson.]

The equation (254) to the curve contains A and A^2. Arranging it as a quadratic in A, we have

$$\left.\begin{array}{r}(4y^5 + 8x^2y^3 + 4x^4y)A^2 + (4y^6 + 10x^2y^4 + 8x^4y^2 + 2x^6 - 4y^4 - 6x^2y^2 - 2x^4)A \\ + y^7 + 4x^2y^5 + 5x^4y^3 + 2x^6y - 2y^5 - 2x^2y^3 + y^3 + 2x^2y = 0.\end{array}\right\} \quad (287)$$

Now, for any values of x and y, two values of A can be found to satisfy the equation. That is, suppose the curve drawn for a given latitude, take some point on it, and this reasoning declares that for some *other latitude*, also, the curve will pass through the chosen point. The fact that the curve returns to its initial shape while the latitude increases to 90° is *prima-facie* evidence that the curve in some way reaches a limit and then returns. This taken in connection with the quadratic character of A points to the *envelope* of the curve as a means of obtaining the limit sought. The envelope is obtained from the condition $B^2 = 4AC$ (in the general quadratic equation); that is, it is the condition that will make A have equal roots in the locus (287).

Squaring the coefficient of A and putting the result equal to four times the coefficient of A^2 into the absolute term and simplifying, we obtain

$$\left.\begin{array}{r}4y^{10}x^2 + 15y^8x^4 + 20y^6x^6 + 10y^4x^8 - x^{12} + 8y^8x^2 + 26y^6x^4 + 30y^4x^6 \\ + 14y^2x^8 + 2x^{10} + 4y^6x^2 + 7y^4x^4 + 2y^2x^6 - x^8 = 0.\end{array}\right\} \quad (288)$$

Factoring,

$$x^2(x^2 + y^2)^2[(x^2 + y^2)(2y + x) + 2y - x][(x^2 + y^2)(2y - x) + 2y + x] = 0. \quad (289)$$

(*m*) The equations given by the several factors of (289) make up the envelope. The first factor, x^2, gives the line Y; the second factor, $(x^2 + y^2)^2$, gives the point (0, 0), the origin: neither is of use to us. But the *remaining factors* give a locus shown in Fig. 5.

[NOTE.—The part for the 3d factor alone will be discussed.]

From which we see that the line ZME may form a boundary within which the curve No. 1

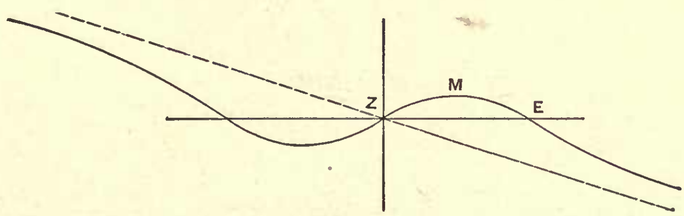


FIG. 5.

may pass through its phases. All facts ascertained respecting the curve develop nothing to conflict with such a conclusion, and all facts are satisfied by such conclusion.

(n) Taking the 3d factor of (289),

$$(x^2+y^2)(2y+x)+2y-x=0. \quad \quad (290)$$

The only infinite branch is given by $\quad 2y+x=0, \quad \quad (291)$

and its asymptote is $\quad y=-\frac{x}{2}, \quad \quad (292)$

since there are no terms of the degree next to the highest.

For points on X and Y,

Put $\quad y=0, \quad x^3-x=0, \quad x(x^2-1)=0, \quad \left.\begin{array}{l} x=0, \text{ origin}; \\ x=\pm 1, \text{ E. and W.} \end{array}\right\} \quad (293)$

Put $\quad x=0, \quad 2y^3+2y=0, \quad y(y^2+1)=0, \quad \left.\begin{array}{l} y=0, \text{ origin}; \\ y=\sqrt{-1}, \text{ imaginary.} \end{array}\right\} \quad (294)$

For tangent at origin, $\quad 2y-x=0, \quad y=\frac{x}{2}. \quad \quad (295)$

For tangent at E. point, put $x+1$ for x.

Lowest-degree terms are $\quad 4y+2x=0; \quad \therefore y=-\frac{x}{2}. \quad \quad (296)$

The curve crosses its asymptote only at the origin.

(o) By (266) the tangent to the curve No. 1 at E is found to be $y=-\dfrac{A}{1+A^2}x$. This expression has a limiting value $\frac{1}{2}$ when $A=1$, $L=45°$. Now this limit of the *inclination of the curve* at E is exactly equal to the *inclination of the envelope* at E by (296).

(p) To find the maximum ordinate of the envelope (for the point M).

The equation to the envelope (290) expanded is

$$2y^3+xy^2+2x^2y+x^3+2y-x=0. \quad \quad (297)$$

$$\frac{dy}{dx}=-\frac{y^2+4xy+3x^2-1}{6y^2+2xy+2x^2+2}.$$

For maximum ordinate we have $\dfrac{dy}{dx}=0$. Putting the numerator $=0$, we have

$$y^2+4xy+3x^2-1=0. \quad \quad (298)$$

Solving (297) and (298) for x and y, we obtain from (298)

$$y = -2x \pm \sqrt{1+x^2}. \quad \ldots \quad (299)$$

Substituting (299) in (297), we have, after squaring and simplifying, the cubic

$$25x^6 + 16x^4 + 12x^2 - 4 = 0; \quad \ldots \quad (300)$$

From which an approximate root is found

$$x^2 = .238126+; \qquad \therefore x = 0.4835422+. \quad \ldots \quad (301)$$

Substituting (301) in (299) we find

$$y = 0.1468784 + \text{for max. ordinate.} \quad \ldots \quad (302)$$

(q) To ascertain the character of the curvature of No. 1 at E.

In (254) move the origin to (1, 0).

The terms of first degree give only the line $y = -\frac{Ax}{1+A^2}$, (266). Taking the terms of second degree in connection with those of the first degree, we have the conic whose curvature near the origin is similar to that of the curve. The equation of the conic is

$$Ay^2 + 8(1+A^2)xy + 9Ax^2 + 2(1+A^2)y + 2Ax = 0. \text{ (hyperbola)}. \quad \ldots \quad (303)$$

If $y = 0$, we have $\qquad 9Ax^2 + 2Ax = 0;$

whence for all values of A, $\quad x = 0, \quad x = -\frac{2}{9}. \quad \ldots \quad (304)$

If $x = 0$, we have $\qquad Ay^2 + 2(1+A^2)y = 0;$

$$y = 0, \qquad y = -\frac{2(1+A^2)}{A}. \quad \ldots \quad (305)$$

The second value of y *always negative* for positive values of A and never less numerically than 4.

For asymptotes,

$$y = \frac{\{-4(A^2+1) - \sqrt{16A^4+23A^2+16}\}\, x - (4A^4+8A^2+5)}{A} \quad \ldots \quad (306)$$

$$y = \frac{\{-4(A^2+1) + \sqrt{16A^4+23A^2+16}\}\, x + (4A^4+6A^2+3)}{A} \quad \ldots \quad (307)$$

(306) and (307) are the asymptotes of the form

$$y = -m_1x - c_1, \qquad y = -m_2x + c_2; \qquad m_1 > m_2, \quad \text{and} \quad c_1 > c_2.$$

These asymptotes always intersect in the 2d angle; and since the curve always goes through the origin and has the two negative intercepts, the curvature at E is always of the form shown in Fig. 6.

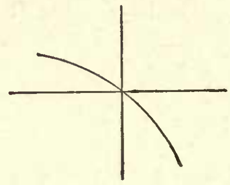


FIG. 6.

(*r*) The conclusions regarding *the envelope* are supported by accurately constructed curves for lat. 30°, 45°, and 60°, and also for a high latitude, $A = 4$; lat. about 75°. For $L < 45°$ the curve is not tangent to the envelope; $L = 45°$ it is tangent at E. point; $L > 45°$ it is tangent at some point S. of the envelope, but the curvature diminishes at E. As L increases, the point of tangency S. moves towards Z, and the curve flattens more and more rapidly until it coincides with the prime-vertical. From a mathematical point of view the latitude to which most significance should be attached would appear to be that at which the curve ceases to swell at E. and begins to swell towards the origin, that is, lat. 45°. But, for our purposes, the latitude that gives the broadest swell to the curve is significant, for the curve will then have the widest departure from the prime-vertical.

The curve constructed with any latitude will have its own greatest ordinate. In the system of curves the maximum of these greatest ordinates must be the maximum ordinate of the envelope; and the curve will be tangent to the envelope at the extremity of this ordinate. The writer has not yet been able to find, analytically, the exact latitude which will give this maximum ordinate.

The approximate numerical value of the latter, and that of the corresponding abscissa (found in (*p*)), when substituted in the equation of the curve expressed as a quadratic in A, (287), will give an approximate value of A, whence the desired latitude.

Assuming the roots equal (the numerical solution verifies the equal roots), we have

$$A = -\left.\frac{(2y^6 + 5x^2y^4 + 4x^4y^2 + x^6 - 2y^4 - 3x^2y^2 - x^4)}{4y(x^2 + y^2)^2}\right]_{\substack{x = .4835422 \\ y = .1468784}} \quad . \quad . \quad . \quad (308)$$

Substituting the values of x and y, the computation gives

$$\left.\begin{aligned} A &= 1.40242 = (\tan L); \\ L &= 54° \ 30' \ 33.'' \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (309)$$

93. *Locus No.* 1. *2d Branch.*

(*a*) By (123), $$y^3 + x^2y + 2Ay^2 + 2Ax^2 - y = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (310)$$

By terms of highest degree $$y(y^2 + x^2) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (311)$$

∴ one infinite branch whose asymptote is parallel to X.

By highest power of x, $\quad y + 2A = 0; \quad \therefore y = -2A$ . . . . (312)

gives asymptote. The curve does not cross its asymptote. This asymptote cuts the meridian, Y, at twice the distance from the origin that the asymptote to the 1st branch of this locus cuts it, for by (256) $y = -A$.

(*b*) To find where the curve cuts Y, put $x = 0$;

$$\therefore y^3 + 2Ay^2 - y = 0, \quad \text{or} \quad y(y^2 + 2Ay - 1) = 0;$$

$$y = 0 \text{ origin, and } y = -A \pm \sqrt{1 + A^2}; \text{ P and P}' \quad \ldots \ldots \quad (313)$$

To find points on X, put $\quad y = 0; \quad \therefore x^2 = 0.$ . . . . (314)

$$\left.\begin{array}{l}\text{Tangent at origin, } y = 0, \text{ axis of X.} \\ \text{Tangents at P and P}', \text{ parallel to X.}\end{array}\right\} \quad \ldots \ldots \quad (315)$$

(*c*) Limiting form for $L = 0$, $\quad A = 0.$

$$y^3 + x^2y - y = 0, \quad \text{or} \quad y(y^2 + x^2 - 1) = 0;$$

$$\left.\begin{array}{l}y = 0, \text{ axis of X, prime-vertical;} \\ y^2 + x^2 = 1, \text{ primitive circle, horizon.}\end{array}\right\} \quad \ldots \ldots \quad (316)$$

(*d*) Limiting form for $L = 90$, $\quad A = \infty.$

$$2Ay^2 + 2Ax^2 = 0, \quad y^2 + x^2 = 0, \text{ origin, zenith.} \quad \ldots \ldots \quad (317)$$

The 3d branch the meridian, axis of Y.

94. *Locus No.* 2. *Alt.-azimuth, error in* L. Articles 44 and 60. Branches—1. Absolute min., the six-hour circle. 2. Absolute max., the meridian.

1st Branch.—Though a great circle, easily constructed, yet, as interesting, a brief analysis is given.

By (126), $\quad y^2 + 2Ay + x^2 - 1 = 0.$ . . . . (318)

There is no infinite branch.

Points on X, $\quad x = \pm 1$, E. and W.; . . . . (319)

Points on Y, $\quad y = -A \pm \sqrt{1 + A^2}$, P and P′. . . . . (320)

Tangents at E. and W., $y = \frac{x}{A}, \quad y = -\frac{x}{A}$. . . . . (321)

Tangents at P and P′ parallel to X.

94a. *Locus No.* 3. *Altitude-azimuth, error in* d. Articles 44 and 61. Branches the same as in No. 2, but the six-hour circle a curve of true max. and min., giving numerical min.

95. *Locus No.* 4. *Time-azimuth, error in* t. Arts. 45 and 62 to 67. Branches—1. Algebraic max. and min., giving numerical min. 2. Absolute min., curve of $q = 90$. 3. Curve of algebraic max. and min., giving generally numerical max., the meridian.

(*a*) By (155), $y^5 + 2x^2y^3 + x^4y + Ay^4 + 2Ax^2y^2 + Ax^4 - Ay^2 - Ax^2 + y = 0$. . (322)

Highest-degree terms, $y(y^2 + x^2)^2 = 0$, ... (323)

gives one infinite branch only, that having an asymptote parallel to X. The coefficient of highest power of x,

$$A + y = 0; \quad \therefore y = -A, \text{ asymptote}, \quad (324)$$

the projection of the parallel of declination, $-d = L$ (see art. 92 (*a*)).

(*b*) Points on X.

If $y = 0$, $x^2(1 - x^2) = 0$, $\left. \begin{array}{l} x^2 = 0, \text{ origin}; \\ x = \pm 1, \text{ E. and W.} \end{array} \right\}$ . . . . (325)

Points on Y.

If $x = 0$, $\left. \begin{array}{l} y(y^4 + Ay^3 - Ay + 1) = 0; \\ y = 0, \text{ origin}; \quad \text{and} \quad y^4 + Ay^3 - Ay + 1 = 0, \quad y = ?; \end{array} \right\}$ . . (326)

(See (*e*) to (*i*).)

(*c*) Tangent at origin, axis of X.

Form of curve at origin, $y = Ax^2$, same as in No. 1 (see art. 92 (*d*)). . . . . . (327)

(*d*) For tangent at E.

$$\left.\frac{dy}{dx}\right]_{\substack{x=1\\y=0}} = \left.\frac{2Ax - 4Ax^3}{x^4 + 1}\right]_{\substack{x=1\\y=0}} = -A; \quad (328)$$

$\therefore$ for tangent at E., similarly, tangent at W., $\left. \begin{array}{l} y = -Ax; \\ y = Ax. \end{array} \right\}$ . . . . (329)

(*e*) For points on Y other than the origin. Returning to (326), we have

$$y^4 + Ay^3 - Ay + 1 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (330)$$

Not having been able directly to factor (330), recourse is had to the equation to the curve on the surface of the sphere, (153), $\cos Z = \frac{A \sin h \cos h}{1 + \sin^2 h}$. The condition required to give points, other than the origin, on Y is $Z = 0$ or $180°$; $\therefore \cos Z = \pm 1$, and we have

$$\pm(1 + \sin^2 h) = A \sin h \cos h. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (331)$$

(*f*) From (330) we see that, with $L = 0$, y is imaginary, and no point on Y; and with $L = 90°$, when $A = \infty$, we have $y(y^2 - 1) = 0$; $y = 0$ and $y = \pm 1$; $\therefore$ the two points on the primitive at its intersections with Y; that is, on the sphere, at the intersections of the meridian with the horizon, giving $h = 0$ for each point.

Now, by (331), for any permissible values of $L < 90°$ to give points on the meridian, sin h, hence h also, must be positive for $\cos Z = +1$; and negative for $\cos Z = -1$; that is, $+y$ cannot exceed unity, and $-y$ cannot be less numerically than unity.

(*g*) Squaring both members of (331), we have

$$1 + 2\sin^2 h + \sin^4 h = A^2 \sin^2 h \cos^2 h = A^2 \sin^2 h(1 - \sin^2 h) = A^2 \sin^2 h - A^2 \sin^4 h, \quad (332)$$

whence $$\sin^4 h + \frac{2 - A^2}{1 + A^2} \sin^2 h = -\frac{1}{1 + A^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (333)$$

Solving (333) as a quadratic,

$$\sin^2 h = \frac{(A^2 - 2) \pm A\sqrt{A^2 - 8}}{2(1 + A^2)}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (334)$$

$$\therefore \sin h = \cos z = \pm \left\{ \frac{A^2 - 2 \pm \sqrt{A^2 - 8}}{2(1 + A^2)} \right\}^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (335)$$

By trig., $\cos z = \frac{1 - \tan^2 \frac{1}{2}z}{1 + \tan^2 \frac{1}{2}z}$, which by the principles of stereographic projection gives

$$\cos z = \frac{1 - y^2}{1 + y^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (336)$$

whence, taking the positive sign of the radical in (335),

(*h*) $$\frac{1 - y^2}{1 + y^2} = \left\{ \frac{A^2 - 2 + A\sqrt{A^2 - 8}}{2(1 + A^2)} \right\}^{\frac{1}{2}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (337)$$

squaring both members of (337), clearing, and dividing by the coefficient of y^4,

$$y^4 - \frac{6A^2 + 2A\sqrt{A^2-8}}{4 + A^2 - A\sqrt{A^2-8}}\,y^2 = -1. \quad \dots \dots \dots \quad (338)$$

Solving (338) as a quadratic,

$$y^2 = \frac{3A^2 + A\sqrt{A^2-8} \pm \{2(A^4 + A^3\sqrt{A^2-8} - A^2 + A\sqrt{A^2-8} - 2)\}^{\frac{1}{2}}}{4 + A^2 - A\sqrt{A^2-8}}; \quad . \quad (339)$$

whence the two values of y, rejecting the inadmissible signs, to conform to the conditions in (f), are

$$y = + \left\{ \frac{3A^2 + A\sqrt{A^2-8} - 2[2(A^4 + A^3\sqrt{A^2-8} - A^2 + A\sqrt{A^2-8} - 2)]^{\frac{1}{2}}}{4 + A^2 - A\sqrt{A^2-8}} \right\}^{\frac{1}{2}} \quad (340)$$

and $$y = - \left\{ \frac{3A^2 + A\sqrt{A^2-8} + 2[2(A^4 + A^3\sqrt{A^2-8} - A^2 + A\sqrt{A^2-8} - 2)]^{\frac{1}{2}}}{4 + A^2 - A\sqrt{A^2-8}} \right\}^{\frac{1}{2}} \quad (341)$$

(i) There remain two other values of y found by taking the negative sign of the radical in (335), whence

$$\frac{1-y^2}{1+y^2} = \pm \left\{ \frac{A^2 - 2 - A\sqrt{A^2-8}}{2(1+A^2)} \right\}^{\frac{1}{2}}; \quad \dots \dots \dots \quad (342)$$

Solving, as in (h), rejecting inadmissible signs, we have

$$y = + \left\{ \frac{3A^2 - A\sqrt{A^2-8} - 2[2(A^4 - A^3\sqrt{A^2-8} - A^2 - A\sqrt{A^2-8} - 2)]^{\frac{1}{2}}}{4 + A^2 + A\sqrt{A^2-8}} \right\}^{\frac{1}{2}} \quad (343)$$

and $$y = - \left\{ \frac{3A^2 - A\sqrt{A^2-8} + 2[2(A^4 - A^3\sqrt{A^2-8} - A^2 - A\sqrt{A^2-8} - 2)]^{\frac{1}{2}}}{4 + A^2 + A\sqrt{A^2-8}} \right\}^{\frac{1}{2}} \quad (344)$$

(k) It is obvious that these values of y are imaginary for any value of $A^2 < 8$ or $A < 2\sqrt{2}$ [also seen from (335)]. Hence, with the lower latitudes, the curve does not touch the meridian at any point other than the origin. For the latitude whose tan is $2\sqrt{2}$ we have two pairs of equal roots, for then (340) and (343) become identical, and (341) and (344) identical. What *have been* in the lower latitudes continuous branches of the curve on each side of the meridian, passing through the zenith and E. and W. points (Fig. 6), at the instant $A = 2\sqrt{2}$, *cross on the meridian.* (Fig. 7, for one pair of equal roots above the horizon.)

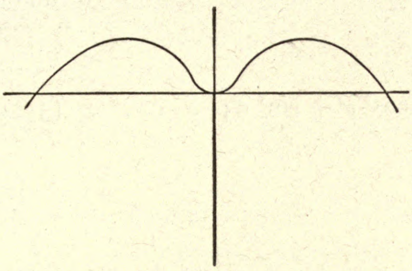


FIG. 6.

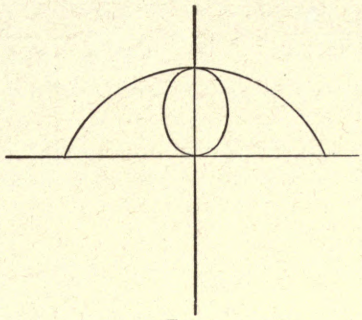


FIG. 7.

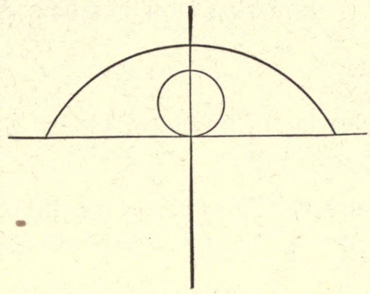


FIG. 8.

Similarly below the horizon for the other pair of equal roots.

But when A becomes greater than $2\sqrt{2}$ by the smallest increment, the equal roots separate, and we have two different roots, and the branches of the curve separate as in Fig. 8.

(l) To investigate for the points on the meridian, Y, where the equal roots occur:

1*st. The altitude of the point.*

In (334) substitute $$A = 2\sqrt{2}, \qquad A^2 = 8;$$

whence $$\sin^2 h = \tfrac{1}{3}, \qquad \sin h = \pm\sqrt{\tfrac{1}{3}} = \pm\frac{1}{\sqrt{3}}. \qquad (345)$$

Hence two branches of the curve cross each other, above the horizon, on the meridian, at an altitude whose sine equals $+\frac{1}{\sqrt{3}}$; the zenith-distance of this point having the *direction towards* the elevated pole. And the lower branches cross on the meridian at a negative altitude numerically equal to the positive, reckoned towards the depressed pole.

(m) 2*d. The declination of the point.*

To find the parallels of declination passing through these points.

Taking the equation in terms of t, L, d (139), the second factor gives

$$\cos^2 t - \frac{2A\cos d}{\sin d}\cos t + \frac{1}{\cos^2 d} - A^2 = 0; \qquad (346)$$

$$\cos t = \pm 1, \quad \text{whence} \quad 1 \mp \frac{2A\cos d}{\sin d} + \frac{1}{\cos^2 d} - A^2 = 0; \qquad (347)$$

$$\therefore \mp\frac{2A\cos d}{\sin d} = A^2 - 1 - \frac{1}{\cos^2 d}. \qquad (348)$$

Substituting $$A = 2\sqrt{2},$$

$$\mp\frac{4\sqrt{2}\cos d}{\sin d} = 7 - \frac{1}{\cos^2 d}. \qquad (349)$$

Squaring both members, $\dfrac{32 \cos^2 d}{\sin^2 d} = 49 - \dfrac{14}{\cos^2 d} + \dfrac{1}{\cos^4 d}$. . . . . (350)

clearing of fractions, $32 \cos^6 d = 49 \sin^2 d \cos^4 d - 14 \sin^2 d \cos^2 d + \sin^2 d$; . . . . (351)

substitute $1 - \cos^2 d$ for $\sin^2 d$,

$$32 \cos^6 d = 49 \cos^4 d - 49 \cos^6 d - 14 \cos^2 d + 14 \cos^4 d + 1 - \cos^2 d; \quad (352)$$

$$\therefore 81 \cos^6 d - 63 \cos^4 d + 15 \cos^2 d - 1 = 0; \quad (353)$$

$$\cos^6 d - \tfrac{7}{9} \cos^4 d + \tfrac{5}{27} \cos^2 d - \tfrac{1}{81} = 0, \quad (354)$$

giving a cubic in $\cos^2 d$.

Roots for $\cos^2 d$ are $\frac{1}{3}$, $\frac{1}{3}$, and $\frac{1}{9}$.

Since d is limited to $\pm 90°$, the negative roots for $\cos d$ are inadmissible. Both of the equal roots $+\sqrt{\frac{1}{3}}$ correspond to a plus declination, and both to a minus declination, numerically equal. Now, this value of $\cos d$ or $\sin p$ is exactly equal to the numerical values of $\sin h$ and $-\sin h$ by (345).

$\therefore \sin p = \sin h$; and, calling p' the polar distance for the negative declination, h' the negative altitude, $\sin p' = \sin h'$ (numerically).

Hence for $A = 2\sqrt{2}$ two branches of the curve cross each other on the meridian at a point midway between the elevated pole and the horizon; and two branches cross at the middle point between the horizon and the depressed pole.

Since $h = p$ it follows that $z = d$, that is, the arc corresponding to $+y$ has the same value as the declination of the body that crosses the meridian at the point where equal roots occur.

Similarly the arc corresponding to $-y$ is numerically the supplement of the negative declination of the body crossing the meridian below the horizon, where the equal roots occur; that is, $z' = 180° - d'$ (numerically).

There remains the root $\frac{1}{3} = \cos d$. This root corresponds to both $+d$ and $-d$, equal numerically and equal to the latitude giving the equal roots. That is, a branch of the curve passes through the zenith and through the nadir. For, since $\cos d = \frac{1}{3}$, $\sin d = \sqrt{1 - \frac{1}{9}}$;

$$\therefore \tan d = 3\sqrt{\tfrac{8}{9}} = \sqrt{8} = 2\sqrt{2}; \quad (355)$$

$$\therefore \tan d = \tan L = A = 2\sqrt{2}.$$

(*n*) Although the expression (326) $y^4 + Ay^3 - Ay + 1 = 0$ may not be factored, approximate roots may be found for different values of A, and the latitude for which equal roots occur may be found as follows, suggested by Lieutenant Rittenhouse:—

The condition for equal roots in a biquadratic equation,

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (356)$$

is $$\{ae - 4bd + 3c^2\}^3 = 27\{ad^2 + eb^2 + c^3 - ace - 2bcd\}, \quad . \quad . \quad . \quad . \quad . \quad (357)$$

and this condition applied to (326) gives $A = 2\sqrt{2}$, as the value of A, for which equal roots occur. Found by substituting in (357),

$$a = 1, \qquad 4b = A, \qquad 6c = 0, \qquad 4d = -A, \qquad e = 1.$$

Substituting this value of A in (326), it will factor, and we obtain

$$\left\{y - \frac{\sqrt{3}-1}{\sqrt{2}}\right\}^2 \left\{y + \frac{\sqrt{3}+1}{\sqrt{2}}\right\}^2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (358)$$

Hence we have *two pairs* of equal roots, and the four roots of the equation are all imaginary for values of A less than $2\sqrt{2}$, and real for values of A greater than $2\sqrt{2}$, but their numerical values are unequal.

These facts show that the curve touches the meridian when $\tan L = 2\sqrt{2}$, and that it has real branches; for, moving the origin to $y = \dfrac{\sqrt{3}-1}{\sqrt{2}}$, we find, for tangents,

$$y = \pm\sqrt{\tfrac{1}{3}}x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (359)$$

These tangents also show how the curve abruptly changes its character in passing a critical value of its parameter A. See Figs. 6, 7, and 8 in (*k*).

(*o*) As a matter of interest the approximate values of the latitude, etc., for the occurrence of equal roots may be found.

Lat. $70°\ 32'$, co-$L = 19°\ 28'$, $\quad h = p = 35°\ 16$, $\quad z = d = 54°\ 44'$,
$\quad -d' = 54°\ 44'$, $\quad z' = (180° - d') = 125°\ 16'$, $\quad -h' = (180° - p') = 35°\ 16'$.

Conclusions, looking at the change in the character of the curve in high latitudes.

In Lat. $70°\ 32'$, the most favorable position of the star whose dec. is $54°\ 44'$ is *on the meridian* at lower culmination, as far as possible from the prime-vertical in bearing. Where this star's parallel of declination crosses the meridian is the point of intersection of *two* curves of algebraic max. and min.; namely, the curve under present discussion and the meridian. It has been asserted that the meridian is generally the locus of *numerical* max.; though the numerical maxima would not be *equal numerically* for the two culminations of the body, yet at both culminations we shall have maxima as compared to the numerical minima on the *first* branch of curve No. 4, at the points of the star's crossing it, for stars whose $\pm d < L$.

This holds good for all such stars with latitudes less than 70° 32′ (approx.). With higher latitudes there may be a belt (a belt above the horizon, also one below) crossing the meridian, within which stars having certain declinations between the limiting declinations of the points at which the two branches of the curve intersect the meridian, will not touch the curve of numerical min. For such stars there will be a numerical max. and a numerical min. on the meridian, corresponding to algebraic max. and min.; for there exists no other minimum with which to compare. From (58) we see that the numerical min. will occur at that culmination of the star that has the less altitude, whether positive or negative. But the *sign* of (58) being negative, the numerical min. will be the algebraic max., regarding the error in t as positive. We therefore see how erroneous may be the statements italicized in arts 5 and 9, when applied to observations in very high latitudes; for, with declinations permitting the star to cross the first branch of the curve, the best position may be very near the meridian in azimuth.

(p) For limiting forms of the curve,

$$L = 90°, \qquad \text{A} = \infty.$$

From (322) we have
$$(x^2 + y^2)(1 - x^2 - y^2) = 0; \qquad (360)$$

$$\left.\begin{aligned} x^2 + y^2 &= 0, \text{ the zenith, a point;} \\ x^2 + y^2 &= 1, \text{ the horizon.} \end{aligned}\right\} \qquad (361)$$

$$L = 0, \qquad \text{A} = 0, \qquad x^4y + 2x^2y^3 + y^5 = 0; \qquad y(x^2 + y^2)^2 = 0; \qquad (362)$$

$$\left.\begin{aligned} y &= 0, \text{ the prime-vertical;} \\ x^2 + y^2 &= 0, \text{ the zenith, a point.} \end{aligned}\right\} \qquad (363)$$

96. *Locus No.* 4. Second branch, $q = 90$, for absolute min.

The same as in Locus No. 1, art. 93.

96*a*. Third branch.—The meridian for algebraic max. and min.; numerical max., for all latitudes below that for which $\tan L = 2\sqrt{2}$, at both culminations of the star; the max. at the culmination having the less altitude, whether positive or negative, being the less. For higher latitudes than $\tan L = 2\sqrt{2}$, *stars within certain declinations* will have one numerical min. and one numerical max. on the meridian, and will not touch the *first branch* (see art. 95 (o)).

97. *Locus No.* 5. *Time-azimuth, error in* L. Arts. 45, 68 to 71. Branches—1. The horizon for absolute min. 2. The meridian for absolute min. 3. The branch given by (179), for algebraic max. and min., giving numerical max.

(a)
$$\left.\begin{aligned} y^6 + x^2y^4 - x^4y^2 - x^6 + 2\text{A}y^5 + 4\text{A}x^2y^3 + 2\text{A}x^4y \\ - 2y^4 - 4x^2y^2 - 2x^4 - 2\text{A}y^3 - 2\text{A}x^2y + y^2 - x^2 = 0. \end{aligned}\right\} \qquad (364)$$

Highest-degree terms give
$$(y - x)(y + x)(y^2 + x^2)^2 = 0. \qquad (365)$$

Hence infinite branches having asymptotes.

To find asymptote parallel to $(y - x) = 0$ we have from (364), retaining only highest-degree terms and those of the next degree,

$$y - x = \left.\frac{-2Ay^5 - 4Ax^2y^3 - 2Ax^4y}{(y+x)(y^2+x^2)^2}\right]_{y=x} = -A; \quad \ldots \ldots \quad (366)$$

Also,

$$y + x = \left.\frac{-2Ay^5 - 4Ax^2y^3 - 2Ax^4y}{(y-x)(y^2+x^2)^2}\right]_{y=-x} = -A; \quad \ldots \ldots \quad (367)$$

$$\therefore y - x = -A \quad \text{and} \quad y + x = -A \text{ for asymptotes.} \quad \ldots \quad (368)$$

There are no parabolic infinite branches.

(*b*) Points on X. Put $y = 0$.

$$-x^6 - 2x^4 - x^2 = 0; \qquad \therefore x^2(x^2+1)^2 = 0; \quad \ldots \ldots \quad (369)$$

whence $x^2 = 0$, origin, zenith, and $(x^2 + 1)^2 = 0$, imaginary. . . . . (370

(*c*) Points on Y. Put $x = 0$.

$$\left.\begin{array}{r} y^6 + 2Ay^5 - 2y^4 - 2Ay^3 + y^2 = 0; \\ y^2(y+1)(y-1)(y^2 + 2Ay - 1) = 0. \end{array}\right\} \quad \ldots \ldots \quad (371)$$

$$\left.\begin{array}{l} y^2 = 0, \text{ origin;} \\ y = \pm 1, \text{ N. and S. points.} \\ y = -A \pm \sqrt{1 + A^2}, \text{ P and P}', \text{ poles.} \end{array}\right\} \quad \ldots \ldots \quad (372)$$

(*d*) For tangents at origin,

$$y^2 - x^2 = 0, \qquad y = x \quad \text{and} \quad y = -x \text{ tangents.} \quad \ldots \ldots \quad (373)$$

(*e*) To find points of the curve on the line $y = x$, put $y = x$ in (364) and we have

$$8Ax^5 - 8x^4 - 4Ax^3 = 0; \quad \ldots \ldots \quad (374)$$

$$x^3(2Ax^2 - 2x - A) = 0 \left\{ \begin{array}{l} x^3 = 0; \\ x = \dfrac{1 \pm \sqrt{1 + 2A^2}}{2A}; \end{array} \right. \quad \ldots \ldots \quad (375)$$

Hence the line $y = x$ cuts the curve *once* at infinity, *three* times at origin, and at *two* other

real points depending for position on A. The six points being thus accounted for, there is no inflexion at the origin.

(f) To find points of the curve on the line $y = -x$, put $y = -x$ in (364), and we obtain

$$-8Ax^5 - 8x^4 + 4Ax^3 = 0; \quad \ldots \ldots \ldots \quad (376)$$

$$\left.\begin{aligned} x^3 &= 0; \\ x &= \frac{-1 \pm \sqrt{1+2A^2}}{2A} \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (377)$$

The same conclusions as in (e).

Also by taking $\cos Z$ equation, (177), and putting $\cos Z = \pm \sqrt{\tfrac{1}{2}}$ (i.e., $y = x$),

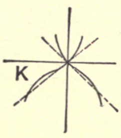


Fig. 8.

we obtain $\qquad \sin h = \pm 1 \qquad$ and $\qquad \sin h = \pm \dfrac{1}{\sqrt{1+2A^2}}$.

Therefore $\qquad h = \pm 90^\circ \qquad$ and $\qquad h = \sin^{-1}\left(\pm \dfrac{1}{\sqrt{1+2A^2}}\right)$.

For the line $y = -x$ we obtain the same value of h as for $y = x$, since $\cos Z = \pm \sqrt{\tfrac{1}{2}}$ corresponds also to $y = -x$.

The results in (e) and (f) indicate for the parts above the horizon the form in Fig. 8, and inflexion occurs at some point K below X.

(g) To find form of curve at N. and S. points.

For N. put $\qquad y = y + 1.$

The coefficient of y is $4A$; always $+$.

The coefficient of x^2 is $(2A - 4)$; $+$ when $A > 2$, $-$ when $A < 2$.

Hence when $A > 2$ the form is shown in Fig. 9; and when $A < 2$, as seen in Fig. 10.

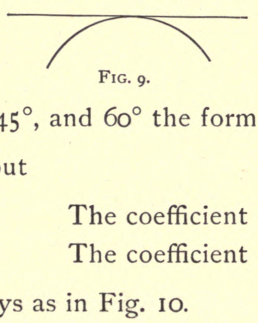


Fig. 9.

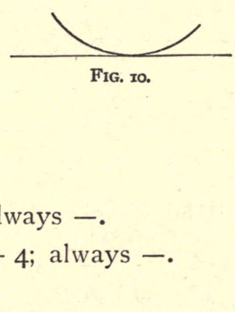


Fig. 10.

For latitudes 30°, 45°, and 60° the form as in Fig. 10.

For S. point put $\qquad y = y - 1.$

The coefficient of y is $-4A$; always $-$.

The coefficient of x^2 is $-2A - 4$; always $-$.

$\therefore$ the form is always as in Fig. 10.

(h) For form at P and P′.

Moving origin to P and P′, $y = y + b$, in which b is $-A + \sqrt{A^2+1}$, for P, and $-A - \sqrt{A^2 - 1}$ for P′, we have,—

Coefficient of y is $\{6b^5 + 10Ab^4 - 8b^3 - 6Ab^2 + 2b\}$.

Coefficient of x^2 is $\{b^4 + 4Ab^3 - 4b^2 - 2Ab - 1\}$.

For latitudes 30°, 45°, and 60°, substituting the values $A = \sqrt{\tfrac{1}{3}}$, 1, and $\sqrt{3}$, and the corresponding values of b, we find form at P and at P′ for all these latitudes, as shown in Fig. 11.

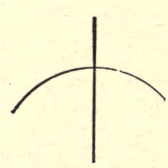


Fig. 11.

(i) To determine where the curve cuts the asymptotes.

In (364) let $y = x - A$, and we obtain

$$(4A^2 + 6)x^4 - (8A^3 + 12A)x^3 + (8A^4 + 8A^2 + 2)x^2 - (4A^5 + 2A^3 - 2A)x - (A^2 - A^6) = 0. \quad (378)$$

When $A = \sqrt{\frac{1}{3}}$, this equation has one positive and one negative root, both less numerically than unity, and has no other real roots; when $A = 1$, one positive root less than unity, one root equal to zero, and no other real roots; when $A = \sqrt{3}$, there are no real roots.

Similarly for the other asymptote, substitute

$$y = -x - A.$$

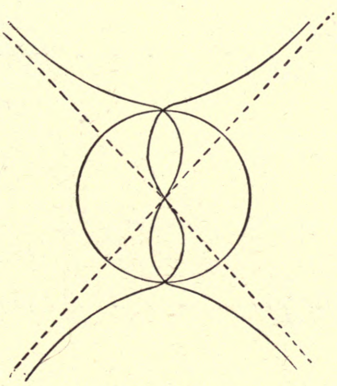
(*k*) For limiting forms of curve.

If $$L = 90, \qquad A = \infty,$$

and by (364), $$(x^2 + y^2)^2 - (x^2 + y^2) = 0, \quad \ldots \quad (379)$$

$x^2 + y^2 = 0$, conjugate point at origin, zenith;
$x^2 + y^2 = 1$, horizon.

If $$L = 0, \qquad A = 0.$$



$$(y^2 - x^2)(y^2 + x^2) - 2(x^2 + y^2)^2 + y^2 - x^2 = 0. \quad \ldots \quad (380)$$

The curve is symmetrical with regard to both X and Y;

tangents at origin, $y = \pm x$;
asymptotes, $y = \pm x$.

The curve crosses asymptotes at origin only, has inflexion at origin and is shown in the accompanying diagram.

98. *Locus No.* 6. *Time-azimuth, error in* d. Arts. 45 and 72 to 75. Branches—1. The meridian for absolute min. 2. The curve of algebraic max. and min., giving numerical max. and min.; given by equation (194).

2d Branch.

(*a*) $$y^4 - x^4 + 2Ay^3 + 2Ax^2y - y^2 + x^2 = 0. \quad \ldots \quad (381)$$

Highest-degree terms give

$$(y - x)(y + x)(y^2 + x^2) = 0; \quad \ldots \quad (382)$$

whence two asymptotes parallel to $y = \pm x$, and there are no parabolic infinite branches.

For asymptote parallel to $y = x$,

$$y - x = \left.\frac{-2Ay(y^2 + x^2)}{(y + x)(y^2 + x^2)}\right]_{y=x} = -A. \quad (383)$$

For asymptote parallel to $y = -x$,

$$y + x = \left.\frac{-2Ay(y^2 + x^2)}{(y - x)(y^2 + x^2)}\right]_{y=-x} = -A; \quad (384)$$

whence $y = x - A$, and $y = -x - A$ are asymptotes. . . . . (385)

(b) Points of curve on X, put $y = 0$.

$$x^4 - x^2 = 0; \qquad \therefore x^2 = 0, \text{ origin, zenith}; \quad (386)$$

$$x = \pm 1, \text{E. and W. points.} \quad (387)$$

Points of curve on Y. Put $x = 0$;

$$y^4 + 2Ay^3 - y^2 = 0; \qquad \left.\begin{aligned} \therefore y^2 &= 0, \text{ zenith}; \\ y &= -A \pm \sqrt{A^2 + 1}, \text{ poles.} \end{aligned}\right\} \quad (388)$$

(c) Tangents at origin, $y = \pm x$.
For tangents at E. and W., differentiating (381),

$$\frac{dy}{dx} = \left.\frac{4x^3 - 2x - 4Axy}{4y^3 - 2y + 2Ax^2 + 6Ay^2}\right]_{\substack{x=\pm 1 \\ y=0}} = \pm \frac{1}{A}; \quad (389)$$

$$\therefore y = \pm \frac{x}{A} \text{ are tangents at E. and W.} \quad (390)$$

(d) To ascertain the form of the curve at P and P', put $y + b$ for y and move origin to $(0, b)$.

The coefficient of y becomes $4b^3 + 6Ab^2 - 2b$. . . . . (391)

The coefficient of x^2 becomes $2Ab + 1$. . . . . (392)

For point P, $\qquad b = \sqrt{A^2 + 1} - A,$

and the coefficient of y is always $+$,

coefficient of x^2 is always $+$;

$\therefore$ at P the form is always as in Fig. 11, art. 97 (h).

For point P′, $$b = -A - \sqrt{A^2 + 1}.$$

The coefficient of y is always $-$.
The coefficient of x^2 is $+$ when $A < \sqrt{\frac{1}{8}}$, giving form . . . . 
The coefficient of x^2 is $-$ when $A > \sqrt{\frac{1}{8}}$, giving form same as at P.

(e) For curve crossing its asymptote. Combine first of (385) and (381). Eliminating y, we have

for which
$$\left.\begin{aligned} x &= \frac{1 + A^2 \pm \sqrt{1 - A^4}}{2A}, \\ y &= \frac{1 - A^2 \pm \sqrt{1 - A^4}}{2A}. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (393)$$

(f) Therefore, when $A < 1$, asymptote crosses curve in two real points;

when $A = 1$, asymptote is tangent to curve (and coincides with the tangent to the curve at E. and W. points);

when $A > 1$, asymptote does not cross the curve.

(g) Combining second of (385) and (381), we find

$$\left.\begin{aligned} x &= \frac{-(1 + A^2) \pm \sqrt{1 - A^4}}{2A}, \\ y &= \frac{1 - A^2 \pm \sqrt{1 - A^4}}{2A}, \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (394)$$

and the same condition as by (f).

(h) For limiting forms of curve.

If $$L = 0, \qquad A = 0,$$

equation (381) becomes $$y^4 - x^4 - y^2 + x^2 = 0; \quad \ldots\ldots\ldots \quad (395)$$

$$(y^2 - x^2)(y^2 + x^2 - 1) = 0. \quad \ldots\ldots\ldots \quad (396)$$

Two lines $y = \pm x$, ($Z = 45^\circ$ and 135°), and primitive circle, horizon.

(i) If $\quad L = 90, \quad A = \infty,$

(381) becomes
$$2Ay^3 + 2Ax^2y = 0; \quad \ldots \quad (397)$$
$$y(y^2 + x^2) = 0; \quad \ldots \quad (398)$$

whence axis of x and conjugate point (zenith).

99. *Locus No. 7. Time-altitude-azimuth, error in* h. Arts. 46 and 76 to 79. Branches —1. The prime-vertical for absolute max. 2. The meridian for absolute min. 3. The horizon for absolute min. 4. For algebraic max. and min., giving a numerical max. by equation (207).

The first three branches are defined.

4th Branch.

By equation (207),

$$y^7 + 2x^2y^5 + x^4y^3 + 2Ay^6 + 6Ax^2y^4 + 6Ax^4y^2 + 2Ax^6 - 2y^5 - 6x^2y^3 - 4x^4y - 2Ay^4 - 4Ax^2y^2 - 2Ax^4 + y^3 = 0. \quad (399)$$

(a) Highest-degree terms, $\quad y^3(y^4 + 2x^2y^2 + x^4); \quad \ldots \quad (400)$

$y^3(y^2 + x^2)^2$ gives no real infinite branches.

The terms $2Ax^6 + y^3x^4$, put equal to zero, indicate a parabolic branch,

$$y^3 = 2Ax^2. \quad \ldots \quad (401)$$

(b) To determine contact of curve with axes. Points on X.

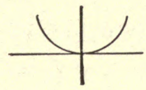
Put
$$y = 0;$$
$$2Ax^6 - 2Ax^4 = 0; \quad \therefore x^4(x^2 - 1) = 0; \quad \ldots \quad (402)$$

Whence
$$x^4 = 0, \text{ zenith}; \quad \ldots \quad (403)$$
$$x = \pm 1, \text{ E. and W.} \quad \ldots \quad (404)$$

Points on Y. Put $\quad x = 0;$

$$\left.\begin{aligned} y^7 + 2Ay^6 - 2y^5 - 2Ay^4 + y^3 = 0; \\ y^3(y^2 - 1)(y^2 + 2Ay - 1) = 0; \end{aligned}\right\} \quad \ldots \quad (405)$$

$$\left.\begin{array}{l} y^3 = 0, \text{ zenith;} \\ y = \pm 1, \text{ N. and S.;} \\ y = -A \pm \sqrt{1 + A^2}, \text{ P and P}'. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (406)$$

For form at origin $y^3 = 2Ax^4$. . . .  . . . . (407)

(*c*) For tangents at E. and W.

$$\left.\frac{dy}{dx}\right]_{(\pm 1.0)} = \left[\frac{-12Ax^5 + 8Ax^3}{-4x^4}\right]_{x=\pm 1} = \pm A; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (408)$$

$y = \pm Ax$, tangents at E. and W. . . . . (409)

(*d*) To determine form of curve at N. and S.

Put $y = y + 1$, and move origin to N.

Coefficient of y is $4A$, always $+$.

Coefficient of x^2 is $2A - 4$; $-$ if $A < 2$; $+$ if $A > 2$.

Hence, when $A < 2$ 

When $A > 2$ 

From next higher terms when $A = 2$, same as $A > 2$.

Put $y - 1$ for y, and move origin to S.

Coefficient of y is $-4A$.
Coefficient of x^2 is $2A + 4$; $\therefore$ at S. always as $A < 2$ for N.

(*e*) To determine the form of curve at P and P′.

Put $y + b$ for y, and move origin to $(0.b)$.

The sign of the coefficient of y is found then to depend on

$$\{7b^4 + 12Ab^3 - 10b^2 - 8Ab + 3\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (410)$$

and the sign of the coefficient of x^2 to depend on

$$\{b^3 + 3Ab^2 - 3b - 2A\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (411)$$

Substituting the values of A and of b for the cases of the curves drawn, lats. 30°, 45°, 60°, we find from the signs of (410) and (411) the form of the curve at P and at P′ to be the same as at N. for A > 2.

The values of A and b being as follows:

For P, since $b = -A + \sqrt{A^2 + 1}$,

$$\text{Lat. } 30°; \; A = \sqrt{\tfrac{1}{3}}, \qquad b = \sqrt{\tfrac{1}{3}};$$

$$\text{Lat. } 45°; \; A = 1, \qquad b = \sqrt{2} - 1;$$

$$\text{Lat. } 60°; \; A = \sqrt{3}, \qquad b = 2 - \sqrt{3}.$$

For P′, since $b = -A - \sqrt{A^2 + 1}$,

$$\text{Lat. } 30°; \; A = \sqrt{\tfrac{1}{3}}, \qquad b = -\sqrt{3};$$

$$\text{Lat. } 45°; \; A = 1, \qquad b = -\sqrt{2} - 1;$$

$$\text{Lat. } 60°; \; A = \sqrt{3}, \qquad b = -\sqrt{3} - 2.$$

(f) To find circles of altitude tangent to locus.

Let

$$x^2 + y^2 = c^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (412)$$

Substitute in (399) $x^2 = c^2 - y^2$ to eliminate x, and we have a cubic in y:

$$y^3 (1 + c^2)^2 - 4yc^4 - 2Ac^4 (1 - c^2) = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (413)$$

Or,

$$y^3 - \frac{4c^4}{(1 + c^2)^2} y - \frac{2Ac^4 (1 - c^2)}{(1 + c^2)^2} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (414)$$

The condition for equal roots is

$$\frac{A^2 c^8 (1 - c^2)^2}{(1 + c^2)^4} = \frac{64 c^{12}}{27 (1 + c^2)^6}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (415)$$

which reduces to
$$A^2(1-c^2)^2 = \frac{64c^4}{27(1+c^2)^2}. \qquad (416)$$

Clearing and arranging in the form of a quadratic in c,

$$c^8 - \frac{54A^2+64}{27A^2}c^4 = -1; \qquad (417)$$

solving,
$$c^4 = \frac{27A^2+32 \pm 8\sqrt{27A^2+16}}{27A^2}, \qquad (418)$$

which always gives *two real* positive values of c^2 and thence always two positive values of c.

(g) To find limiting forms of locus,

$$L = 90, \qquad A = \infty,$$

in (399)
$$2y^6 + 6x^2y^4 + 6x^4y^2 + 2x^6 - 2y^4 - 4x^2y^2 - 2x^4 = 0; \qquad (419)$$

$$(y^2+x^2)^2(x^2+y^2-1) = 0; \qquad (420)$$

whence peculiar point
$$(y^2+x^2) = 0, \text{ zenith}; \qquad (421)$$

$$x^2+y^2 = 1, \text{ circle, horizon}. \qquad (422)$$

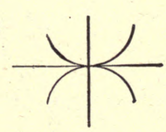
(h)
$$L = 0, \qquad A = 0,$$

in (399)
$$\left.\begin{aligned} y^7 + 2x^2y^5 + x^4y^3 - 2y^5 - 6x^2y^3 - 4x^4y + y^3 &= 0; \\ y^6 + 2x^2y^4 + x^4y^2 - 2y^4 - 6x^2y^2 - 4x^4 + y^2 &= 0. \end{aligned}\right\} \qquad (423)$$

Highest-degree terms $y^2(y^2+x^2)^2 = 0$, gives infinite branches whose asymptotes, parallel to X, by coefficient of highest power of x are

$$y^2 - 4 = 0, \qquad y = \pm 2. \qquad (424)$$

For form at origin
$$y^2 = 4x^4;$$
$$y = \pm 2x^2. \qquad (425)$$



(i) For contact with X and Y.

If $y = 0$,
$$4x^4 = 0, \text{ at origin only, zenith}. \qquad (426)$$

If $x = 0$,
$$\left.\begin{aligned} y^6 - 2y^4 + y^2 &= 0; \\ y^2(y^2-1)^2 &= 0. \end{aligned}\right\} \qquad (427)$$

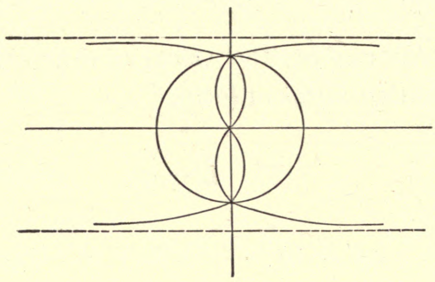
$$\left.\begin{array}{ll} y^2 = & 0, \text{ origin;} \\ y = & \pm 1, \text{ double points.} \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (428)$$

(*k*) For form at (0, $\pm$ 1) move origin, and lowest-degree terms give

$$\left.\begin{array}{r} y^2 - x^2 = 0; \\ \text{tangents,} \qquad y = \pm x. \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (429)$$

(*l*) Combining (424) with (423), we find that the curve does not cross its asymptotes.

Form when A = 0.



100. *Locus No.* 8. *Time-altitude-azimuth, error in* t. Arts. 46 and 80 to 83. Branches —1. Absolute max., the prime-vertical. 2. Absolute min., the six-hour circle. 3. Algebraic max. and min., giving numerical max. and min., the meridian. We have, also, the equator for a branch giving a constant error.

Though all branches are great circles, and therefore directly defined in the projection, yet it may be of interest to analyze the equation to the equator; that of the six-hour circle having been treated in art. 94.

The equation to the equator, by (223), is

$$y^2 + x^2 - \frac{2}{A}y - 1 = 0. \quad \ldots\ldots\ldots\ldots \quad (430)$$

From $y^2 + x^2$, there is no infinite branch.

Contact with X, put
$$y = 0,$$
$$x^2 = \pm 1, \text{ E. and W.}$$

Contact with Y, put
$$x = 0,$$
$$y^2 - \frac{2}{A}y = 1; \quad \ldots\ldots\ldots\ldots \quad (431)$$

$$\therefore y = \frac{1 \pm \sqrt{1 + A^2}}{A}. \quad \ldots\ldots\ldots\ldots \quad (432)$$

For tangents at E. and W., move origin to $(\pm 1, 0)$,

$$y = \mp Ax, \text{ tangents at E. and W.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (433)$$

Limiting forms,

$$L = 0, \qquad A = 0, \qquad y = 0, \text{ axis of X, the prime-vertical;}$$
$$L = 90^\circ, \qquad A = \infty, \qquad y^2 + x^2 = 1, \text{ primitive circle, horizon.}$$

101. *Locus No.* 9. *Time-altitude-azimuth, error in* d. Arts. 46 and 84. Branches—**1.** The meridian for absolute min. 2. The prime-vertical for absolute max. 3. The curve of elongations, $q = 90^\circ$, for algebraic max. and min. giving numerical min.

The third branch has been treated in art. 58. It is noteworthy that this curve, $q = 90^\circ$, though it has occurred frequently in the loci preceding No. 9, yet in no case heretofore given has it been the locus of *algebraic* max. and min.

102. *Locus No.* 10. *Time-azimuth and altitude-azimuth, for error in* L, *giving the same numerical error in the computed azimuth.* Arts. 85 to 90. Branches—1. Errors equal, having signs alike, the prime-vertical and the curve of $q = 90$. 2. Errors equal numerically with contrary signs.

2d Branch.

By equation (253),

$$y^4 + 3x^2y^2 + 2x^4 + 2Ay^3 + 2Ax^2y - y^2 - 2x^2 = 0. \quad . \quad . \quad . \quad . \quad . \qquad (434)$$

(*a*) Highest-degree terms have no real factors; therefore, there is no infinite branch. No real factors in terms of lowest degree; hence, no branches through the origin.

(*b*) For contact with X and Y.

Put $$y = 0, \qquad 2x^4 - 2x^2 = 0, \qquad x^2(x^2 - 1) = 0; \quad . \quad . \quad . \quad . \quad . \quad . \qquad (435)$$

$$\left.\begin{array}{l} x^2 = 0, \text{ conjugate point at origin, zenith;} \\ x = \pm 1, \text{ E. and W.} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (436)$$

Put $$x = 0, \qquad y^4 + 2Ay^3 - y^2 = 0, \qquad \text{or} \qquad y^2(y^2 + 2Ay - 1) = 0; \quad . \quad . \quad . \qquad (437)$$

$$y^2 = 0, \text{ conjugate point at origin;} \qquad y = -A \pm \sqrt{1 + A^2}, \text{ P and P}'. \quad . \quad . \qquad (438)$$

(*c*) To find form of curve at E. and W.

Moving origin to (1, 0) in (434). Terms of lowest degree,

$$2Ay = -4x,$$

$$\left.\begin{array}{lr} \text{give} & y = \dfrac{-2x}{A} \text{ for tangent at E.;} \\ \text{similarly} & y = \dfrac{2x}{A} \text{ for tangent at W.} \end{array}\right\} \quad \text{(439)}$$

Numerical values of $\dfrac{y}{x}$ for different latitudes:

$$3.47 \text{ in lat. } 30^\circ, \qquad 2 \text{ in lat. } 45^\circ, \qquad 1.15 \text{ in lat. } 60^\circ. \quad \text{(440)}$$

(*d*) For forms at P and P′.

For P, move origin by substituting

$$y - A + \sqrt{A^2 + 1} \text{ for } y.$$

The coefficient of y is found to be $2(A^2 + 1 - 2A\sqrt{A^2+1})\sqrt{A^2+1}$, always $+$.

The coefficient of x^2 is found to be $4A^2 + 1 - 4A\sqrt{A^2+1}$,

which is $+$ when $A < \frac{1}{4}\sqrt{2}$ and $-$ when $A > \frac{1}{4}\sqrt{2}$.

For the curves drawn for lat. 30°, 45°, and 60°, the smallest value of A is $\frac{1}{3}\sqrt{3}$ (lat. 30°), which is greater than $\frac{1}{4}\sqrt{2}$; hence, for the latitudes used, the coefficient of x^2 is in all cases *minus* and the curve at P is of the form .

For P′, substitute $y - A - \sqrt{A^2+1}$ for y, and we find

the coefficient of y to be always $-$,

and the coefficient of x^2 to be always $+$;

hence, form at P′ always as shown for P above.

(*e*) For limiting forms of curve.

If $\qquad L = 0, \qquad A = 0,$

$$\left.\begin{array}{lr} \text{equation (434) becomes} & y^4 + 3x^2y^2 + 2x^4 - y^2 - 2x^2 = 0, \\ \text{or} & (y^2 + x^2 - 1)(y^2 + 2x^2) = 0. \end{array}\right\} \quad \text{(441)}$$

$$\left.\begin{array}{l} y^2 + x^2 = 1, \text{ primitive circle, horizon;} \\ (y^2 + 2x^2) = 0, \text{ conjugate point at origin, zenith.} \end{array}\right\} \quad \text{(442)}$$

If $\qquad L = 90^\circ, \qquad A = \infty,$

the equation becomes $2y^3 + 2x^2y = 0$, or $y(y^2 + x^2) = 0$; . . . . (443)

$$\left.\begin{array}{r} y = 0, \text{ giving X;} \\ y^2 + x^2 = 0, \text{ conjugate point at origin.} \end{array}\right\} \quad \text{. . . . .} \quad (444)$$

103. Though this treatise is called Azimuth, and though an attempt has been made to present the subject in a new light on some points, no attempt has been made to put into the hands of the worker in the field the practical methods of observation for the elimination of errors. The Coast Survey office, in its appendices * to the reports, leaves nothing to be desired on the score of precision in work with refined instruments. For the navigator, provided with the sextant, there is, it is hoped, something gained by the investigations pursued, applicable as well to more refined work. Even with the sextant, when the observation on shore for determining the direction of the meridian cannot be taken at the most favorable instant, errors in the data may sometimes be eliminated by observing at the same relative points both east and west of the meridian. Even though the error by the observation on one side of the meridian may be large, the mean of the results on both sides may leave no resulting error in the computed azimuth.

But the azimuth problem has also been particularly useful in studying the variations of the astronomical triangle. For this, the problems on time and on latitude do not bring out so many truths. Inspection of the terms in which the errors are expressed usually suffices in the two problems mentioned. Not so, often, in the azimuth problem, though much will be obvious.

104. A single instance will suffice. Taking the case $dZ = -\dfrac{\cos q \cos d}{\cos h}\, dt$ (58) for error in Z, due to a positive error in the hour angle, which, by art. 34, employing the general astronomical triangle, is error in time. Having regard to the signs of trigonometric functions and to the resulting sign before the whole expression necessitates a knowledge of the approximate values of the particular parts of the triangle found in the expression within the limits of quadrants. We look to the resulting sign of dZ to determine whether it is an algebraic max. or min. at particular points. If $+$ and dZ changes from an increasing function to a decreasing function, we shall have an algebraic max. If $+$, and the change is from a decreasing function to an increasing function, we shall have an algebraic min. If $-$ and dZ changes from a numerically increasing function to a decreasing function, we shall have an algebraic min., though a numerical maximum negative. If $-$ and change is from a numerically decreasing function to an increasing function, we shall have an algebraic max., but a numerical min.

From (58), by inspection alone, we cannot tell whether there is any point, other than on the meridian, where the function changes from increasing to decreasing or the contrary; but if there be such a point for $+ d < L$, for instance, we know from the change that occurs on the meridian at upper culmination that here will be an algebraic min. (numerical max. neg.); and at the point of change before lower culmination we shall have an algebraic max. (numerical min. negative); at lower culmination an algebraic min. (numerical max. negative), not having, however, the same numerical value as at upper transit, the latter being greater; at the next point east of the meridian an algebraic max. (numerical min. negative). The curve

* By Charles A. Schott.

passing through W, Z, E, as found in No. 4, determines for a given declination whether the star reaches any point on it; then we can discriminate between the algebraic max. and min. If there is a point, we have already seen how to discriminate. If there is no point on the curve, as for certain declinations, when the observer is in very high latitudes, then we have the meridian alone to consider; d being less than L, at upper culmination we shall have an algebraic min. (numerical max. negative), and at lower culmination an algebraic max. (numerical min. negative), for cos h will be greater for lower transit; hence dZ numerically less but negative.

105. In respect to the elimination of errors by observing at symmetrically situated points on both sides of the meridian, notwithstanding a poor point of observation for one side alone:—In No. 1, No. 2, and No. 3, if the body is observed at the same altitude on both sides of the meridian, the error due to error in h, that due to error in L, and that due to error in d will be eliminated. Therefore, since in No. 1, even though the error in h should not be eliminated, yet if the star is observed at $q = 90°$ the error will reduce to zero, we should endeavor to observe at that point on both sides of the meridian; therefore select a close circumpolar star. But if the star crosses the prime-vertical we should observe at its crossing the curve of min. errors for error in h on both sides of the meridian.

In No. 4 the error in t cannot be eliminated by the mean of the results, but if $q = 90°$ for $d > L$, it will be zero, and if $d < L$ there is a point of min. error; this point, then, or $q = 90°$, should be selected on both sides of the meridian, for then, by No. 5 and No. 6, the errors in L and d will be eliminated. Therefore the value of a close circumpolar star to apply these conditions.

APPENDIX.

A Scheme for Deriving Curves from the Variations of the Spherical Triangle.

1. The curves already obtained from the problems on azimuth suggest the possibility of obtaining a great variety of loci derived from the variations of the spherical triangle; and, from the astronomical triangle, by regarding always L and d constant in finding the maximum and minimum effects of variation in any part, due to the variation in any other part, we may find many interesting curves, though the number will be less than if not restricted to L and d constant.

2. The ultimate equation in x and y may be disengaged or not, at one's pleasure, from any idea of the sphere, since A (or B, or C) will be a single arbitrary constant (see art. 55); but the sphere furnishes the astronomical triangle as the means of obtaining the equations. Therefore there may be curves of interest to the mathematician, though having no utility.

3. To adopt a system for obtaining the equations to the loci, we have the following:

1st. Any four parts of the triangle involved, and any three of these given, the remaining part may be found.

2d. Form all possible groups of four parts.

3d. In each group, each part may be taken in turn as the one to find.

4th. In every case, the partial differential coefficient, for each *given part* in error and the *part to be found*, may be taken.

5th. Each partial differential coefficient* will have some kind of max. and min., as defined for the purposes of this treatise; assuming always, in determining these max. and min., that L and d are constant, in order to conform to the nature of the case in the problems used in practice, even though L or d, or both, may have been variable in the problem giving the partial differential.

6th. If *algebraic max. and min.* occur, the differential coefficient's *own* first differential put equal to zero will show their existence from the equations derived, *all* parts except L and d varying.

7th. If there were an instrument to measure q accurately, and if our instrument for measuring Z gives precision, and if, in nature, there were problems demanding Z and q as given parts, then all possible problems based on the astronomical triangle might be of utility.

8th. Lacking utility, all possible cases will still give examples in curve-tracing.

9th. Great circles of the sphere will probably recur many times in the loci of the preceding article, as they have done in the curves Nos. 1 to 10 defined in this treatise; and possibly the same novel loci that those curves give, or that may be found in the future, will also recur.

10th. Making the combinations, we have 90 individual cases; each one having a reciprocal, which need not be considered, thus making 180.

* Excepting when it consists of functions of the constants, only.

11th. In these ninety cases are included the equations to curves found in this treatise, and equations for additional useful problems (time; latitude; computation of the altitude in the problem of astronomical bearing; and the time-altitude-latitude-azimuth, in distinction from time-altitude-declination-azimuth, called in this work time-altitude-azimuth).

12th. In making the cases for partial differentials, the two constants will, of course, be given parts in the original problem, and either one of the variables will be another given part; the remaining variable, a part to be found.

13th. For illustration:

1st A, b, and c, being given, to find B. A and b constant, c and B variable, we shall indicate by writing $A, b, \frac{dB}{dc}$; $\frac{dB}{dc}$, being the coefficient of error in B due to a small error in c. For max. and min. effects, we have $d\left(\frac{dB}{dc}\right) = 0$, regarding b and c (corresponding to co-L and co-d in the companion triangle), always as constant in this part of the work.

2d. For the reciprocal:

Given A, b, and B, to find c. A and b constant, B and c variable, $A, b, \frac{dc}{dB}$, error in c, due to error in B.

The reciprocal will, in all cases, be omitted as giving the same locus as the first form, but having max. and min. interchanged; and to simplify the indication we shall omit the symbol d, writing only $A, b, \frac{B}{c}$.

14th. For combinations of four of the six parts—A, B, C, a, b, c—of the triangle we have

1.	A, B, C, a.	(6)
2.	A, B, C, b.	(7)
3.	A, B, C, c.	(8)
4.	A, B, a, b.	(3)
5.	A, B, a, c.	(4)
6.	A, B, b, c.	(2)
7.	A, C, a, b.	(9)
8.	A, C, a, c.	(10)
9.	A, C, b, c.	(11)
10.	A, a, b, c.	(5)
11.	B, C, a, b.	(12)
12.	B, C, a, c.	(13)
13.	B, C, b, c.	(14)
14.	B, a, b, c.	(1)
15.	C, a, b, c.	(15)

The numbers (1), (2), and (3) show the parts taken for the curves described in this treatise. (4), the parts for the *time-altitude-latitude-azimuth*; and (5) those for the *time-sight*, the problem of *latitude at any time*, and the *computation* of h for use in the astronomical bearing.

15th. From the astronomical triangle,

$$A = t, \quad B = Z, \quad C = q, \quad a = \text{co-}h, \quad b = \text{co-}d, \quad c = \text{co-}L;$$

we derive the following cases for loci:

(1) Parts involved, B, a, b, c; Z, h, d, L.

$a, b, \frac{B}{c}$; $h, d, \frac{Z}{L}$: alt.-az., error in L. . . . . (*a*)

$a, c, \frac{B}{b}$; $h, L, \frac{Z}{d}$: alt.-az., error in d. . . . . (*b*)

$b, c, \frac{B}{a}$; $h, L, \frac{Z}{h}$: alt.-az., error in h. . . . . (*c*)

$B, a, \frac{b}{c}$; $Z, h, \frac{d}{L}$. . . . . (*d*)

$B, b, \frac{a}{c}$; $Z, d, \frac{h}{L}$. . . . . (*e*)

$B, c, \frac{a}{b}$; $Z, L, \frac{h}{d}$. . . . . (*f*)

(2) Parts involved, A, B, b, c; t, Z, d, L.

$A, b, \frac{B}{c}$; $t, d, \frac{Z}{L}$: time-az., error in L. . . . . (*g*)

$A, c, \frac{B}{b}$; $t\ L, \frac{Z}{d}$: time az., error in d. . . . . (*h*)

$b, c, \frac{B}{A}$; $d, L, \frac{Z}{t}$: time-az., error in t. . . . . (*i*)

$B, b, \frac{A}{c}$; $Z, d, \frac{t}{L}$. . . . . (*k*)

$B, c, \frac{A}{b}$; $Z, L, \frac{t}{d}$. . . . . (*l*)

$A, B, \frac{b}{c}$; $t, Z, \frac{d}{L}$. . . . . (*m*)

(3) Parts involved A, B, a, b; t, Z, h, d.

$A, a, \frac{B}{b}$; $t, h, \frac{Z}{d}$: time-alt.-dec.-az., error in d. . . . . . (n)

$A, b, \frac{B}{a}$; $t, d, \frac{Z}{h}$: time-alt.-dec.-az., error in h. . . . . . (o)

$a, b, \frac{B}{A}$; $h, d, \frac{Z}{t}$: time-alt.-dec.-az., error in t. . . . . . (p)

$A, B, \frac{a}{b}$; $t, Z, \frac{h}{d}$. . . . . (q)

$B, a, \frac{A}{b}$; $Z, h, \frac{t}{d}$. . . . . (r)

$B, b, \frac{A}{a}$; $Z, d, \frac{t}{h}$. . . . . (s)

(4) Parts involved, A, B, a, c; t, Z, h, L.

$A, a, \frac{B}{c}$; $t, h, \frac{Z}{L}$: time-alt.-lat.-az., error in L. . . . . . . (t)

$A, c, \frac{B}{a}$; $t, L, \frac{Z}{h}$: time-alt.-lat.-az., error in h. . . . . . . (u)

$a, c, \frac{B}{A}$; $h, L, \frac{Z}{t}$: time-alt.-lat.-az., error in t. . . . . . . (v)

$A, B, \frac{a}{c}$; $t, Z, \frac{h}{L}$. . . . . (w)

$B, a, \frac{A}{c}$; $Z, h, \frac{t}{L}$ . . . . (x)

$B, c, \frac{A}{a}$; $Z, L, \frac{t}{h}$. . . . . (y)

(5) Parts involved, A, a, b, c; t, h, d, L.

$a, b, \frac{A}{c}$; $h, d, \frac{t}{L}$: { time-sight, error in L. / reciprocal, for lat. problem, error in t. } . . . . (A)

$a, c, \frac{A}{b}$; $h, L, \frac{t}{d}$: time-sight, error in d. . . . . (B)

$b, c, \frac{A}{a}$; $d, L, \frac{t}{h}$: { time-sight, error in h. / recip., alt. computed, error in t. } . . . . . (C)

$A, a, \frac{c}{b}$; $t, h, \frac{L}{d}$: lat. problem, error in d. . . . . (D)

$A, b, \frac{c}{a}$; $t, d, \frac{L}{h}$: { lat. problem, error in h. / recip., alt. computed, error in L. } . . . . (E)

$A, c, \frac{a}{b}$; $t, L, \frac{h}{d}$: alt. computed, error in d. . . . . (F)

(6) Parts involved, A, B, C, a; t, Z, q, h.

$A, B, \frac{C}{a}$; $t, Z, \frac{q}{h}$. . . . . (G)

$A, C, \frac{B}{a}$; $t, q, \frac{Z}{h}$. . . . . (H)

$A, a, \frac{B}{C}$; $t, h, \frac{Z}{q}$. . . . . (I)

$B, C, \frac{A}{a}$; $Z, q, \frac{t}{h}$. . . . . (K)

$B, a, \frac{A}{C}$; $Z, h, \frac{t}{q}$. . . . . (L)

$C, a, \frac{A}{B}$; $q, h, \frac{t}{Z}$. . . . . (M)

(7) Parts involved, A, B, C, b; t, Z, q, d.

$A, B, \frac{C}{b}$; $t, Z, \frac{q}{d}$. . . . . (N)

$A, C, \frac{B}{b}$; $t, q, \frac{Z}{d}$. . . . . (O)

$A, b, \frac{B}{C}$; $t, d, \frac{Z}{q}$. . . . . (P)

$B, C, \frac{A}{b}$; $Z, q, \frac{t}{d}$. . . . . (Q)

$B, b, \frac{A}{C}$; $Z, q, \frac{t}{q}$. . . . . (R)

$C, b, \frac{A}{B}$; $q, d, \frac{t}{Z}$. . . . . (S)

(8) Parts involved, A, B, C, c; t, Z, q, L.

$A, B, \frac{C}{c}$; $t, Z, \frac{q}{L}$. . . . . (T)

$A, C, \frac{B}{c}$; $t, q, \frac{Z}{L}$. . . . . (U)

$A, c, \frac{B}{C}$; $t, L, \frac{Z}{q}$. . . . . (V)

$B, C, \frac{A}{c}$; $Z, q, \frac{t}{L}$. . . . . (W)

$B, c, \frac{A}{C}$; $Z, L, \frac{t}{q}$. . . . . (X)

$C, c, \frac{A}{B}$; $q, L, \frac{t}{Z}$. . . . . (Y)

(9) Parts involved, A, C, a, b; t, q, h, d.

$A,\ C,\ \frac{a}{b};\quad t,\ q,\ \frac{h}{d}.$ . . . . (a)

$A,\ a,\ \frac{C}{b};\quad t,\ h,\ \frac{q}{d}.$ . . . . (b)

$A,\ b,\ \frac{C}{a};\quad t,\ d,\ \frac{q}{h}.$ . . . . (c)

$C,\ a,\ \frac{A}{b};\quad q,\ h,\ \frac{t}{d}.$ . . . . (d)

$C,\ b,\ \frac{A}{a};\quad q,\ d,\ \frac{t}{h}.$ . . . . (e)

$a,\ b,\ \frac{A}{C};\quad h,\ d,\ \frac{t}{q}.$ . . . . (f)

(10) Parts involved, A, C, a, c; t, q, h, L.

$A,\ C,\ \frac{a}{c};\quad t,\ q,\ \frac{h}{L}.$ . . . . (g)

$A,\ a,\ \frac{C}{c};\quad t,\ h,\ \frac{q}{L}.$ . . . . (h)

$A,\ c,\ \frac{C}{a};\quad t,\ L,\ \frac{q}{h}.$ . . . . (i)

$C,\ a,\ \frac{A}{c};\quad q,\ h,\ \frac{t}{L}.$ . . . . (k)

$C,\ c,\ \frac{A}{a};\quad q,\ L,\ \frac{t}{h}.$ . . . . (l)

$a,\ c,\ \frac{A}{C};\quad h,\ L,\ \frac{t}{q}.$ . . . . (m)

(11) Parts involved, A, C, b, c; t, q, d, L.

$A,\ C,\ \frac{b}{c};\quad t,\ q,\ \frac{d}{L}.$ . . . . (n)

$A,\ b,\ \frac{C}{c};\quad t,\ d,\ \frac{q}{L}.$ . . . . (o)

$A,\ c,\ \frac{C}{b};\quad t,\ L,\ \frac{q}{d}.$ . . . . (p)

$C,\ b,\ \frac{A}{c};\quad q,\ d,\ \frac{t}{L}.$ . . . . (q)

$C,\ c,\ \frac{A}{b};\quad q,\ L,\ \frac{t}{d}.$ . . . . (r)

$b,\ c,\ \frac{A}{C};\quad d,\ L,\ \frac{t}{q}.$ . . . . (s)

(12) Parts involved, $B, C, a, b;\ Z, q, h, d$.

$$B, C, \frac{a}{b};\ Z, q, \frac{h}{d}. \quad \ldots\ldots \quad (t)$$

$$B, a, \frac{C}{b};\ Z, h, \frac{q}{d}. \quad \ldots\ldots \quad (u)$$

$$B, b, \frac{C}{a};\ Z, d, \frac{q}{h}. \quad \ldots\ldots \quad (v)$$

$$C, a, \frac{B}{b};\ q, h, \frac{Z}{d}. \quad \ldots\ldots \quad (w)$$

$$C, b, \frac{B}{a};\ q, d, \frac{Z}{h}. \quad \ldots\ldots \quad (x)$$

$$a, b, \frac{B}{C};\ h, d, \frac{Z}{q}. \quad \ldots\ldots \quad (y)$$

(13) Parts involved, $B, C, a, c;\ Z, q, h, L$.

$$B, C, \frac{a}{c};\ Z, q, \frac{h}{L}. \quad \ldots\ldots \quad (A)$$

$$B, a, \frac{C}{c};\ Z, h, \frac{q}{L}. \quad \ldots\ldots \quad (B)$$

$$B, c, \frac{C}{a};\ Z, L, \frac{q}{h}. \quad \ldots\ldots \quad (C)$$

$$C, a, \frac{B}{c};\ q, h, \frac{Z}{L}. \quad \ldots\ldots \quad (D)$$

$$C, c, \frac{B}{a};\ q, L, \frac{Z}{h}. \quad \ldots\ldots \quad (E)$$

$$a, c, \frac{B}{C};\ h, L, \frac{Z}{q}. \quad \ldots\ldots \quad (F)$$

(14) Parts involved, $B, C, b, c;\ Z, q, d, L$.

$$B, C, \frac{b}{c};\ Z, q, \frac{d}{L}. \quad \ldots\ldots \quad (G)$$

$$B, b, \frac{C}{c};\ Z, d, \frac{q}{L}. \quad \ldots\ldots \quad (H)$$

$$B, c, \frac{C}{b};\ Z, L, \frac{q}{d}. \quad \ldots\ldots \quad (I)$$

$$C, b, \frac{B}{c};\ q, d, \frac{Z}{L}. \quad \ldots\ldots \quad (K)$$

$$C, c, \frac{B}{b};\ q, L, \frac{Z}{d}. \quad \ldots\ldots \quad (L)$$

$$b, c, \frac{B}{C};\ d, L, \frac{Z}{q}. \quad \ldots\ldots \quad (M)$$

(15) Parts involved, C, a, b, c; q, h, d, L.

$$C,\ a,\ \frac{b}{c};\ q,\ h,\ \frac{d}{L}. \quad \text{(N)}$$

$$C,\ b,\ \frac{a}{c};\ q,\ d,\ \frac{h}{L}. \quad \text{(O)}$$

$$C,\ c,\ \frac{a}{b};\ q,\ L,\ \frac{h}{d}. \quad \text{(P)}$$

$$a,\ b,\ \frac{C}{c};\ h,\ d,\ \frac{q}{L}. \quad \text{(Q)}$$

$$a,\ c,\ \frac{C}{b};\ h,\ L,\ \frac{q}{d}. \quad \text{(R)}$$

$$b,\ c,\ \frac{C}{a};\ d,\ L,\ \frac{q}{h}. \quad \text{(S)}$$

4. Fundamental equations from trigonometry for the solution of the various problems of finding any part of the triangle, three parts being given; needed for transforming, in finding equations to the curve of max. and min. variations, in any three terms; and used, if desired, for differentiating directly for the general expression of error in one part due to an error in some other part. For convenience, these equations are written here, and the parts involved written at the left hand; (1), (2), etc., conforming to the groups given in the 14th and 15th sections of art. 3.

$B, a, b, c.$	$\cos b = \cos c \cos a + \sin c \sin a \cos B.$	(1)
$A, B, b, c.$	$\sin A \cot B = \sin c \cot b - \cos c \cos A.$	(2)
$A, B, a, b.$	$\sin a \sin B = \sin b \sin A.$	(3)
$A, B, a, c.$	$\sin B \cot A = \sin c \cot a - \cos c \cos B.$	(4)
$A, a, b, c.$	$\cos a = \cos c \cos b + \sin c \sin b \cos A.$	(5)
$A, B, C, a.$	$\cos A = -\cos B \cos C + \sin B \sin C \cos a.$	(6)
$A, B, C, b.$	$\cos B = -\cos C \cos A + \sin C \sin A \cos b.$	(7)
$A, B, C, c.$	$\cos C = -\cos A \cos B + \sin A \sin B \cos c.$	(8)
$A, C, a, b.$	$\sin C \cot A = \sin b \cot a - \cos b \cos C.$	(9)
$A, C, a, c.$	$\sin c \sin A = \sin a \sin C.$	(10)
$A, C, b, c.$	$\sin A \cot C = \sin b \cot c - \cos b \cos A.$	(11)
$B, C, a, b.$	$\sin C \cot B = \sin a \cot b - \cos a \cos C.$	(12)
$B, C, a, c.$	$\sin B \cot C = \sin a \cot c - \cos a \cos B.$	(13)
$B, C, b, c.$	$\sin b \sin C = \sin c \sin B.$	(14)
$C, a, b, c.$	$\cos c = \cos a \cos b + \sin a \sin b \cos C.$	(15)

5. Additional formulas involving five parts, useful in transforming.

$A, B, a, b, c.$ $\sin a \cos B = \sin c \cos b - \cos c \sin b \cos A.$ . . . . (16)

$B, C, a, b, c.$ $\sin b \cos C = \sin a \cos c - \cos a \sin c \cos B.$ . . . . (17)

$A, C, a, b, c.$ $\sin c \cos A = \sin b \cos a - \cos b \sin a \cos C.$ . . . . (18)

$A, C, a, b, c.$ $\sin a \cos C = \sin b \cos c - \cos b \sin c \cos A.$ . . . . (19)

$A, B, a, b, c.$ $\sin b \cos A = \sin c \cos a - \cos c \sin a \cos B.$ . . . . (20)

$B, C, a, b, c.$ $\sin c \cos B = \sin a \cos b - \cos a \sin b \cos C.$ . . . . (21)

$A, B, C, a, b.$ $\sin A \cos b = \sin C \cos B + \cos C \sin B \cos a.$ . . . . (22)

$A, B, C, b, c.$ $\sin B \cos c = \sin A \cos C + \cos A \sin C \cos b.$ . . . . (23)

$A, B, C, a, c.$ $\sin C \cos a = \sin B \cos A + \cos B \sin A \cos c.$ . . . (24)

$A, B, C, a, c.$ $\sin A \cos c = \sin B \cos C + \cos B \sin C \cos a.$ . . . (25)

$A, B, C, a, b.$ $\sin B \cos a = \sin C \cos A + \cos C \sin A \cos b.$ . . . . (26)

$A, B, C, b, c.$ $\sin C \cos b = \sin A \cos B + \cos A \sin B \cos c.$ . . . . (27)

6. The following differential equations will furnish all the partial differential coefficients required, without recourse to differentiating the equation of the particular problem considered:

$A, a, b, c.$ $da = \cos C\,db + \cos B\,dc + \sin b \sin C\,dA.$ . . . . (28)

$B, a, b, c.$ $db = \cos A\,dc + \cos C\,da + \sin c \sin A\,dB.$ . . . . (29)

$C, a, b, c.$ $dc = \cos B\,da + \cos A\,db + \sin a \sin B\,dC.$ . . . . (30)

Or, corresponding,

$t, h, d, L.$ $dh = \cos q\,dd + \cos Z\,dL - \cos d \sin q\,dt.$ . . . . . . (31)

$Z, h, d, L.$ $dd = \cos t\,dL + \cos q\,dh - \cos L \sin t\,dZ.$ . . . . . . (32)

$q, h, d, L.$ $dL = \cos Z\,dh + \cos t\,dd - \cos h \sin Z\,dq.$ . . . . . . (33)

7. LOCI IN TIME-ALTITUDE-LATITUDE-AZIMUTH.

Given t, h, L, *to find* Z;

A, a, c, *to find* B. Group (4)

1st. Error in *t*. $a, c, \dfrac{B}{A}.$

$$\sin B \cot A = \sin c \cot a - \cos c \cos B. \qquad \text{Eq. (4)}$$

$$\sin Z \cot t = \cos L \tan h - \sin L \cos Z.$$

From (28), $$0 = \cos C\,db + \sin b \sin C\,dA. \qquad (34)$$

From (29), $$db = \sin c \sin A\,dB. \qquad (35)$$

Eliminating db, $$\frac{dB}{dA} = \frac{\sin b \sin C}{\sin c \sin A \cos C} = \frac{\sin b}{\sin a \cos C}. \qquad (36)$$

$$\frac{dZ}{dt} = \frac{\cos d}{\cos h \cos q}. \qquad (37)$$

It is obvious that the curve of elongations is that of absolute max.

Putting d (2d member) $= 0$, eliminating dh and dq, and simplifying, we obtain
$$\sin Z(\cos d \cos q \sin h + \cos L \cos Z) = 0. \qquad (38)$$

$\sin Z = 0$, the meridian one branch.

The other branch by $$\sin b \cos C \cos a + \sin c \cos B = 0. \qquad (39)$$

To turn into B, a, c, substitute from (17) $\sin b \cos C$,

$$\sin a \cos a \cos c - \cos^2 a \sin c \cos B + \sin c \cos B = 0. \qquad (40)$$

Substitute $$\cos^2 a = 1 - \sin^2 a.$$

$$\sin c \sin^2 a \cos B = - \sin a \cos a \cos c. \qquad (41)$$

$$\cos B = - \cot a \cot c. \qquad (42)$$

$$\cos Z = - \tan h \tan L. \quad \text{Equator.} \qquad (43)$$

No novel curve, and (43) proves to be a curve of constant error; for, when $d = 0$, by substituting (19) in (37) we have $$\frac{dZ}{dt} = \frac{1}{\sin L}.$$

Meridian for algebraic max. and min. gives numerical min. at that culmination of the star having the less altitude, and numerical max. for greater altitude. But both these are numerical min. for stars whose $\pm d > L$, as compared with absolute max. for $q = 90°, 270°$. The equator divides the meridian into branches of algebraic max. and min.

2d. Error in L. $t, h, \frac{Z}{L}$; $A, a, \frac{B}{C}$.

By (28) and (29), $$db = \cos A\,dc + \sin c \sin A\,dB; \qquad (44)$$

$$0 = \cos C\,db + \cos B\,dc. \qquad (45)$$

Eliminating db,

$$\cos C \cos A\,dc + \cos C \sin c \sin A\,dB + \cos B\,dc = 0; \qquad (46)$$

$$\frac{dB}{dc} = - \frac{\cos C \cos A + \cos B}{\cos C \sin c \sin A};$$

and by (7), $$= - \frac{\sin C \sin A \cos b}{\cos C \sin c \sin A}. \qquad (47)$$

or $$\frac{dZ}{dL} = \frac{\tan q \sin d}{\cos L}. \qquad (48)$$

By inspection, $q = 90$, curve of elongation, absolute max.;

$q = 0$, meridian, absolute min.;

$Z = 90$, prime vertical, algebraic max. and min., giving numerical max.

No novel curve.

3d. Error in h. $t, L, \frac{Z}{h}; A, c, \frac{B}{a}$.

By (28) and (29), $$da = \cos C\, db; \qquad (49)$$

$$db = \cos C\, da + \sin c \sin A\, dB; \qquad (50)$$

$$\therefore \frac{dB}{da} = \frac{\sin^2 C}{\sin c \sin A \cos C} = \frac{\sin C \tan C}{\sin c \sin A} = \frac{\tan C}{\sin a}; \qquad (51)$$

$$\frac{dZ}{dh} = -\frac{\tan q}{\cos h} = -\frac{\sin q}{\cos q \cos h}. \qquad (52)$$

By inspection, the meridian for absolute min. and curve of elongations for absolute max. Algebraic max. and min. by

$$d\left(-\frac{\sin q}{\cos q \cos h}\right) = 0. \qquad (53)$$

$$\cos^2 q \cos h\, dq + \sin^2 q \cos h\, dq + \cos q \sin q \sin h\, dh = 0; \qquad (54)$$

$$\cos h\, dq + \cos q \sin q \sin h\, dh = 0.$$

Substitute $dh = \tan Z \cos h\, dq,$ and $\sin q = \frac{\sin Z \cos L}{\cos d}$:

$$\cos d \cos Z + \cos q \sin^2 Z \cos L \sin h = 0. \qquad (55)$$

As an example, we shall give the work instead of merely indicating steps.

$$\sin b \cos B + \cos C \sin^2 B \sin c \cos a = 0. \qquad (56)$$

Substitute $\sin^2 B = 1 - \cos^2 B$:

$$\sin b \cos B + \cos C \sin c \cos a - \cos C \sin c \cos a \cos^2 B = 0. \qquad (57)$$

Substitute $\cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b}$, from (15),

and clearing,

$$\left.\begin{aligned} &\sin^2 b \sin a \cos B + \sin c \cos c \cos a - \sin c \cos^2 a \cos b \\ &\quad - \sin c \cos a \cos c \cos^2 B + \sin c \cos^2 a \cos b \cos^2 B = 0. \end{aligned}\right\} \qquad (58)$$

Substitute $$1 - \cos^2 b = \sin^2 b:$$

$$\left.\begin{array}{r}\sin a \cos B - \sin a \cos^2 b \cos B + \sin c \cos c \cos a - \sin c \cos^2 a \cos b \\ - \sin c \cos a \cos c \cos^2 B + \sin c \cos^2 a \cos b \cos^2 B = 0.\end{array}\right\} \quad . \quad (59)$$

Substitute $$\cos b = \cos c \cos a + \sin c \sin a \cos B,$$

and $$\cos^2 b = \cos^2 c \cos^2 a + 2 \sin c \sin a \cos c \cos a \cos B + \sin^2 c \sin^2 a \cos^2 B,$$

and we have

$$\left.\begin{array}{l}\sin a \cos B - \sin a \cos^2 c \cos^2 a \cos B - 2 \sin^2 a \sin c \cos c \cos a \cos^2 B \\ \qquad - \sin^3 a \sin^2 c \cos^3 B + \sin c \cos c \cos a - \sin c \cos^3 a \cos c \\ \qquad - \sin^2 c \sin a \cos^2 a \cos B - \sin c \cos c \cos a \cos^2 B \\ \qquad + \sin c \cos c \cos^3 a \cos^2 B = 0.\end{array}\right\} \quad . \quad (60)$$

Collecting coefficients of powers of cos B,

$\cos^3 B$	$\cos^2 B$	$\cos B$	
$- \sin^3 a \sin^2 c$	$- 2 \sin^2 a \sin c \cos c \cos a$	$+ \sin a$	$+ \sin c \cos c \cos a$
$+ \sin^2 c \sin a \cos^2 a$	$- \sin c \cos a \cos c$	$- \sin a \cos^2 c \cos^2 a$	$- \sin c \cos^3 a \cos^2 c$
	$+ \sin c \cos^3 a \cos c$	$- \sin^2 c \sin a \cos^2 a$	

$$\left.\begin{array}{r}\sin^2 c \sin a (\cos^2 a - \sin^2 a) \cos^3 B - 3 \sin c \cos c \cos a \sin^2 a \cos^2 B \\ + \sin^3 a \cos B + \sin c \cos c \cos a \sin^2 a = 0.\end{array}\right\} \quad . \quad . \quad (61)$$

Dividing by $\sin a$,

$$\left.\begin{array}{r}\sin^2 c (\cos^2 a - \sin^2 a) \cos^3 B - 3 \sin c \cos c \cos a \sin a \cos^2 B + \sin^2 a \cos B \\ + \sin c \cos c \cos a \sin a = 0;\end{array}\right\} \quad . \quad . \quad (62)$$

$$\left.\begin{array}{r}\cos^2 L (\sin^2 h - \cos^2 h) \cos^3 Z - 3 \cos L \sin L \sin h \cos h \cos^2 Z + \cos^2 h \cos Z \\ + \cos L \sin L \sin h \cos h = 0.\end{array}\right\} \quad . \quad . \quad (63)$$

Let B represent $\cos L$, whence $\sqrt{1 - B^2}$ will represent $\sin L$.
Then, from article (55) preceding,

$$B^2 \left\{ \frac{(1 - r^2)^2}{(1 + r^2)^2} - \frac{4r^2}{(1 + r^2)^2} \right\} \frac{y^3}{r^3} - 3B \sqrt{1 - B^2} \times \frac{2r}{1 + r^2} \times \frac{1 - r^2}{1 + r^2} \times \frac{y^2}{r^2}$$

$$+ \frac{4r^2}{(1 + r^2)^2} \times \frac{y}{r} + B \sqrt{1 - B^2} \times \frac{1 - r^2}{1 + r^2} \times \frac{2r}{1 + r^2} = 0. \quad . \quad . \quad . \quad (64)$$

G. C. D., $r^3 (1 + r^2)^2$;

$$\left.\begin{array}{l}\therefore B^2 (1 - 6r^2 + r^4) y^3 \\ \qquad - 3B \sqrt{1 - B^2} \times 2r^2 (1 - r^2) y^2 \\ \qquad + 4r^4 y \\ \qquad + B \sqrt{1 - B^2} (1 - r^2) 2r^4 = 0.\end{array}\right\} = \left\{\begin{array}{l}B^2 y^3 - 6B^2 r^2 y^3 + B^2 r^4 y^3 \\ - 6B \sqrt{1 - B^2} r^2 y^2 + 6B \sqrt{1 - B^2} \times r^4 y^2 \\ + 4r^4 y \\ + 2B \sqrt{1 - B^2} \times r^4 - 2B \sqrt{1 - B^2} \times r^6 = 0.\end{array}\right. \quad (65)$$

Substituting $\quad r^2 = x^2 + y^2, \qquad r^4 = x^4 + 2x^2y^2 + y^4,$

and $\quad r^6 = x^6 + 3x^4y^2 + 3x^2y^4 + y^6,$

and we obtain,

$$\left.\begin{array}{l} B^2y^3 - 6B^2y^3x^2 - 6B^2y^5 + B^2y^3x^4 + 2B^2x^2y^5 + B^2y^7 - 6B\sqrt{1-B^2}y^2x^2 \\ \quad - 6B\sqrt{1-B^2}y^4 + 6B\sqrt{1-B^2}y^2x^4 + 12B\sqrt{1-B^2}x^2y^4 \\ \quad + 6B\sqrt{1-B^2}y^6 + 4x^4y + 8x^2y^3 + 4y^5 + 2B\sqrt{1-B^2}x^4 \\ \quad + 4B\sqrt{1-B^2}x^2y^2 + 2B\sqrt{1-B^2}y^4 - 2B\sqrt{1-B^2}x^6 - 6B\sqrt{1-B^2}x^4y^2 \\ \quad - 6B\sqrt{1-B^2}x^2y^4 - 2B\sqrt{1-B^2}y^6 = 0. \end{array}\right\} \quad (65)$$

Collecting coefficients of y^7, y^6, etc.,

$$\begin{array}{c|c|c|c|c|c|c} y^7 & y^6 & y^5 & y^4 & y^3 & y^2 & y \\ B^2 & +6B\sqrt{1-B^2} & -6B^2 & -6B\sqrt{1-B^2} & +B^2 & -6B\sqrt{1-B^2}x^2 & +4x^4 \\ & -2B\sqrt{1-B^2} & +2B^2x^2 & +12B\sqrt{1-B^2}x^2 & -6B^2x^2 & +6B\sqrt{1-B^2}x^4 & \\ & & +4 & +2B\sqrt{1-B^2} & +B^2x^4 & +4B\sqrt{1-B^2}x^2 & \\ & & & -6B\sqrt{1-B^2}x^2 & +8x^2 & -6B\sqrt{1-B^2}x^4 & \end{array} \quad (66)$$

absolute terms, $\quad + 2B\sqrt{1-B^2}x^4, \qquad - 2B\sqrt{1-B^2}x^6,$

reducing,

$$\left.\begin{array}{l} B^2y^7 + 4B\sqrt{1-B^2}y^6 - 6B^2y^5 + 2B^2x^2y^5 + 4y^5 - 4B\sqrt{1-B^2}y^4 \\ \quad + 6B\sqrt{1-B^2}x^2y^4 + B^2y^3 - 6B^2x^2y^3 + B^2x^4y^3 + 8x^2y^3 \\ \quad - 2B\sqrt{1-B^2}x^2y^2 + 4x^4y + 2B\sqrt{1-B^2}x^4 - 2B\sqrt{1-B^2}x^6 = 0. \end{array}\right\} \quad (67)$$

Dividing by the coefficient of y^7, and arranging in order of terms of highest degree,

$$\left.\begin{array}{l} y^7 + 2x^2y^5 + x^4y^3 + \dfrac{4\sqrt{1-B^2}}{B}y^6 + \dfrac{6\sqrt{1-B^2}}{B}x^2y^4 - \dfrac{2\sqrt{1-B^2}}{B}x^6 \\ \quad - 6y^5 + \dfrac{4}{B^2}y^5 - 6x^2y^3 + \dfrac{8}{B^2}x^2y^3 + \dfrac{4}{B^2}x^4y \\ \quad - \dfrac{4\sqrt{1-B^2}}{B}y^4 - \dfrac{2\sqrt{1-B^2}}{B}x^2y^2 + \dfrac{2\sqrt{1-B^2}}{B}x^4 + y^3 = 0, \end{array}\right\} \quad (68)$$

which is the equation to the stereographic projection of the locus of algebraic max. and min. for error in h in time-alt.-lat.-azimuth.

8. TIME-SIGHT.

Given h, L, *and* d, *to find* t.

$$\left.\begin{array}{l} \cos a = \cos b \cos c + \sin b \sin c \cos A\,; \\ \sin h = \sin d \sin L + \cos d \cos L \cos t\,; \end{array}\right\} \quad (69)$$

from which are derived the well-known formulas,

$$\sin \tfrac{1}{2}t = \sqrt{\frac{\cos s \sin (s - h)}{\cos L \sin p}}, \quad \text{in which} \quad s = \tfrac{1}{2}(L + p + h); \quad . \quad . \quad . \quad (70)$$

$$\cos \tfrac{1}{2}t = \sqrt{\frac{\sin s \sin (s - z)}{\sin \text{co-}L \sin p}}, \quad \text{in which} \quad s = \tfrac{1}{2}(\text{co-}L + p + h); \quad . \quad . \quad (71)$$

$$\tan \tfrac{1}{2}t = \sqrt{\frac{\cos s \sin (s - h)}{\sin (s - L) \cos (s - p)}}, \quad \text{in which} \quad s = \tfrac{1}{2}(L + h + p). \quad . \quad (72)$$

Small errors in the data will have the same effect on the hour-angle computed by each of these formulas. So far as inexactness in the logarithmic tables is concerned, (72) will give the result nearest to precision.

If $t > 90°$, (71) will be preferable to (70).
If $t < 90°$, (70) will be preferable to (71).

9. 1*st Case.* Error in t owing to error in h.

Group (5), $\quad d, L, \frac{t}{h}; \; b, c, \frac{A}{a}.$

Differentiating (69), $\quad - \sin a \, da = - \sin b \sin c \sin A \, dA \; ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$

$$\frac{dA}{da} = \frac{\sin a}{\sin A \sin b \sin c} = \frac{\sin b}{\sin B \sin b \sin c} = \frac{1}{\sin B \sin c}; \quad . \quad . \quad . \quad . \quad (74)$$

$$\frac{dt}{dh} = - \frac{1}{\sin Z \cos L} = - \frac{1}{\sin q \cos d} = - \frac{\cos h}{\sin t \cos d \cos L}. \quad . \quad (75)$$

Inspection of (75) shows that for a small error in h the most favorable position of the body is either on the prime vertical or at elongation, whichever is attained; hence the nearer in bearing to the prime vertical the better. At a given place, all bodies that cross the prime vertical, if seized exactly on it, will give the same value to the numerical min. whatever the declination; and the less the latitude the better.

If the body does not cross the prime vertical the best position is when at elongation; and the less the declination the better. For a given body, seized exactly at $q = 90°$, or $270°$, all latitudes permissible ($L < d$) will give the same value to the numerical min.

For $- d$ the most favorable position is, theoretically, in the horizon.

If observed at the same distance from the meridian, in bearing, on both sides, the constant error in h (instrumental) will be eliminated; therefore the value of the method of equal-altitudes (with a correction for change of declination when there is any change).

The most unfavorable position is on the meridian giving absolute max.

10. Inspection gives all that is needed, but we may easily deduce the algebraic max. and min., giving the numerical min. This is interesting, as giving both the curve of elongations and the prime vertical; and since we do not need the equation for curve-tracing, let it be found in terms of t, L, d.

From (75), since L and d are constant,

$$d\left(-\frac{\cos h}{\sin t}\right) = 0; \quad \ldots \quad (76)$$

$$\sin t \sin h \, dh + \cos h \cos t \, dt = 0; \quad \ldots \quad (77)$$

$$\therefore \frac{dt}{dh} = -\frac{\sin t \sin h}{\cos t \cos h} = \frac{-\cos h}{\sin t \cos d \cos L} \text{ (by 75)}; \quad \ldots \quad (78)$$

$$\therefore \sin^2 t \sin h \cos L \cos d - \cos t \cos^2 h = 0. \quad \ldots \quad (79)$$

From trig. substitute, $\quad 1 - \sin^2 h \quad$ for $\quad \cos^2 h,$

and $\quad \sin L \sin d + \cos L \cos d \cos t \quad$ for $\quad \sin h.$

Performing the operations and collecting terms,

$$\left.\begin{array}{r}\sin L \sin d \cos L \cos d \cos^2 t + (\cos^2 L \cos^2 d + \sin^2 L \sin^2 d - 1) \cos t \\ + \cos L \cos d \sin L \sin d = 0.\end{array}\right\} \ldots \quad (80)$$

For coefficient of $\cos t$ substitute

$$-\sin^2 L - \cos^2 L \quad \text{for} \quad -1,$$

and we obtain $\quad [\cos^2 L (\cos^2 d - 1) + \sin^2 L (\sin^2 d - 1)] \cos t;$

$$\therefore [-\cos^2 L \sin^2 d - \sin^2 L \cos^2 d] \cos t.$$

Substituting this in (80), and dividing by the coefficient of $\cos^2 t$,

$$\cos^2 t - \left\{\frac{\tan d}{\tan L} + \frac{\tan L}{\tan d}\right\} \cos t = -1. \quad \ldots \quad (81)$$

Solving (81) as a quadratic,

$$\cos^2 t - \{\quad\} \cos t + \tfrac{1}{4}\left(\frac{\tan d}{\tan L} + \frac{\tan L}{\tan d}\right)^2 = \tfrac{1}{4}\left(\frac{\tan^2 d}{\tan^2 L} + 2 + \frac{\tan^2 L}{\tan^2 d}\right) - 1$$

$$= \tfrac{1}{4}\left(\frac{\tan^2 d}{\tan^2 L} - 2 + \frac{\tan^2 L}{\tan^2 d}\right); \quad \ldots \quad (82)$$

$$\cos t - \tfrac{1}{2}\left(\frac{\tan d}{\tan L} + \frac{\tan L}{\tan d}\right) = \pm \tfrac{1}{2}\left(\frac{\tan d}{\tan L} - \frac{\tan L}{\tan d}\right). \quad \ldots \quad (83)$$

Taking upper sign of second member,

$$\cos t = \frac{\tan a}{\tan L}, \text{ which is for } Z = 90^\circ \text{ when } d < L. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

Taking lower sign,

$$\cos t = \frac{\tan L}{\tan d}, \text{ for } q = 90^\circ \text{ when } d > L. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

Though in the problems of azimuth both of these have been found to be branches of numerical min. at the same time, yet both have never been derived algebraically from the given expression for error.

11. *2d Case.*—Error in t owing to error in L.

$$h, d, \frac{t}{L}; \ a, b, \frac{A}{c}.$$

From (69) we obtain

$$0 = -\cos b \sin c \, dc + \sin b \cos c \cos A \, dc - \sin b \sin c \sin A \, dA. \quad . \quad . \quad . \quad (86)$$

Dividing by $\sin b$
$$\frac{dA}{dc} = -\frac{\cot b \sin c - \cos c \cos A}{\sin c \sin A}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

And from (2),
$$= -\frac{\sin A \cot B}{\sin c \sin A} = -\frac{\cot B}{\sin c}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

$$\therefore \frac{dt}{dL} = \frac{1}{\tan Z \cos L} \qquad dt_L = \frac{dL}{\tan Z \cos L}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

Inspection shows that the prime vertical and the curve of elongations form the locus of minimum errors in the computed time; the former for absolute min. when $\pm d < L$, and the latter for algebraic max. and min., giving numerical min., when $\pm d > L$. It will not be necessary to derive the equation to the latter. The lower the latitude the better.

12. *3d Case.*—Error in t owing to error in d.

$$h, L, \frac{t}{d}; \ a, c, \frac{A}{b}.$$

Interchanging b and c, d and L in the preceding, we derive

$$dt_d = \frac{dd}{\tan q \cos d}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

Inspection shows that the locus of numerical min. consists of the same branches as for error in L; but here the curve of elongation gives absolute min. when $\pm d > L$; and the prime-vertical gives numerical min. from algebraic max. and min., for $\pm d < L$.

$$\text{Total error, } dt = -\frac{1}{\sin Z \cos L}dh + \frac{1}{\tan Z \cos L}dL + \frac{1}{\tan q \cos d}dd. \quad . \quad . \quad . \quad . \quad (91)$$

13. An erroneous assertion by Chauvenet should be corrected.

Chauvenet's Astronomy, vol. i., page 212, says:

$$\text{“}dt = \frac{dd}{\cos d \tan q}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

which shows that the error

[1] in the declination of a given star produces the least effect when the star is on the prime vertical;

[2] and of different stars the most eligible is that which is nearest the equator.”

(*a*) Now, as far as error in declination is concerned, if we may *select* the star to be *given*, we should take one having $d > L$ (which will not come on the prime vertical) and observe at greatest elongation, $q = 90°$, when $dt_d = 0$. If observed at this instant, it matters not what the declination may be, between the limits $d = 90°$ and $d > L$; but, for failure to seize the star exactly at $q = 90°$, the less the declination the better.

(*b*) Supposing the star given us does cross the prime vertical, then [1] is true but [2] is false.

(*c*) Since errors in d are less likely to occur than errors in L and h, and since *these* two errors will give the least dt when $Z = 90°$, we would better select a star that crosses the prime vertical, when $dt_h = \operatorname{cosec} Z \sec L\, dh = \sec L\, dh$, and $dt_L = \sec L \cot Z\, dL = 0$.

(*d*) Therefore, for error in h and error in L it matters not what the declination is, provided the star is observed on the prime vertical.

(*e*) But for error in d, the most eligible star of those that cross the prime vertical is that having *greatest* declination,—not as asserted in [2] preceding.

(*f*) Therefore, for errors in all the *data*, theoretically, the most eligible star is the one crossing the prime vertical, that has the greatest declination; that is, $d = L$, making $Z = 90°$ and $q = 90°$ when the star is on the meridian. Practically we should avoid stars on the meridian and take one whose $d < L$.

14. *Summary and Proof of Fallacy.*

1st. The most favorable star when the error in declination alone is concerned is *any one* whose $d > L$, provided it is observed at $q = 90°$, then $dt_d = 0$. This excludes stars that cross the prime vertical.

2d. But Chauvenet makes his *given* star come on the prime vertical. Therefore $d < L$,

and q cannot $= 90°$; the error dt_d cannot reduce to 0, but it will be least for q nearest to $90°$, which will occur when $Z = 90°$; that is, when the star is on the prime vertical.

3d. Of several given stars observed on the prime vertical the most favorable one will be that whose $\cos d \tan q$, when on the prime vertical, is greatest. And this will be for $d = L$. For notwithstanding the greater d is, the less the $\cos d$, yet the greater the d the nearer will q be to $90°$, and $\tan q$ will be the greater; and $\tan q$ will preponderate, making $\cos d$ by its side insignificant.

Singularly enough, in (92) Chauvenet gives $\cos d$ overwhelming preponderance over $\tan q$, provided the star is on the prime vertical; whereas in $dZ = \frac{1}{\tan q \cos h} dh$, of like form, he permits $\cos h$ to have no influence, and assigns the star to the prime vertical on the strength of $\tan q$ alone.

Proof of Fallacy.

Changing the form of Chauvenet's expression into one equivalent, we have, since $\sin Z \cos L = \sin q \cos d$,

$$dt = \frac{\cos q}{\sin Z \cos L} dd. \qquad (93)$$

Therefore, for *any given star* the nearer q and Z both are to $90°$ the more favorable; and as one nears $90°$ so does the other.

For any one of *several given stars* restricted to $d < L$ and to observations on the prime vertical, $(Z = 90°)$,

$$dt = \frac{\cos q}{\cos L} dd. \qquad (94)$$

L being fixed, to select the best one of these stars we have to consider q alone, and take that one giving q nearest to $90°$ which will make $\cos q$ the least.

Now, since $$\sin q = \frac{\sin Z \cos L}{\cos d},$$

when $Z = 90°$, $$\sin q = \frac{\cos L}{\cos d}. \qquad (95)$$

Cos L being constant, q will be nearest to $90°$ when the declination is greatest; or directly from (95) $q = 90°$ when $d = L$, and the star that is *farthest* from the equator, instead of *nearest* (as asserted by Chauvenet), is the most eligible.

15. LATITUDE PROBLEM.

Latitude by an altitude observed at any time.
Given t, h, *and* d, *to find* L.
Given A, a, *and* b, *to find* c.

$$\left.\begin{aligned} \cos a &= \cos b \cos c + \sin b \sin c \cos A\,; \\ \sin h &= \sin d \sin L + \cos d \cos L \cos t\,; \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (96)$$

from which are derived the well-known formulas,

$$\left.\begin{aligned} \tan \phi &= \tan d \sec t\,; \\ \cos \phi' &= \frac{\sin \phi \sin h}{\sin d}\,; \\ L &= \phi \mp \phi'. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (97)$$

Attending to the algebraic signs of ϕ and ϕ', there will still be two values of L. The proper value will be determined by the known approximate value.

16. 1*st Case.* Error in L owing to error in h.

$$t,\, d,\, \frac{L}{h}\,;\; A,\, b,\, \frac{c}{a}.$$

From (96), A and b being constant,

$$-\sin a\, da = -\cos b \sin c\, dc + \sin b \cos c \cos A\, dc\,; \quad \ldots\ldots \quad (98)$$

$$\frac{da}{dc} = \frac{\cos b \sin c - \sin b \cos c \cos A}{\sin a}\,; \quad \ldots\ldots\ldots \quad (99)$$

and from (16),

$$= \frac{\sin a \cos B}{\sin a} = \cos B\,; \quad \ldots\ldots\ldots\ldots \quad (100)$$

$$\frac{dc}{da} = \frac{1}{\cos Z}\,; \qquad dL_h = \frac{dh}{\cos Z}. \quad \ldots\ldots\ldots\ldots \quad (101)$$

By inspection, the most favorable position of the body is on the meridian, and the least favorable on the prime vertical, giving absolute max. If the star does not come on the prime vertical, the least favorable position will be on the curve of elongations, $q = 90^\circ, 270^\circ$. The latter condition is obvious, but it may be found algebraically, there being algebraic max. and min.

17. *2d Case.* Error in L owing to error in t.

This is the reciprocal of (89),

$$\therefore \frac{dL}{dt} = \tan Z \cos L, \qquad (102)$$

and the max. and min. are interchanged, as compared with art. 11.

18. *3d Case.* Error in L owing to error in d.

$$t, h, \frac{L}{d}; A, a, \frac{c}{b}.$$

From (96),

$$0 = -\cos c \sin b\, db - \cos b \sin c\, dc + \cos b \sin c \cos A\, db + \sin b \cos c \cos A\, dc; \qquad (103)$$

$$\frac{dc}{db} = -\frac{\sin b \cos c - \cos b \sin c \cos A}{\sin c \cos b - \cos c \sin b \cos A}; \qquad (104)$$

and by (16) and (19),

$$= -\frac{\sin a \cos C}{\sin a \cos B} = -\frac{\cos C}{\cos B}; \qquad (105)$$

$$\therefore dL_d = -\frac{\cos q}{\cos Z} dd. \qquad (106)$$

The meridian is found to be the locus of algebraic max. and min., and this gives numerical max. for $\pm d > L$ and numerical min. for $\pm d < L$, since for $q = 90°$ there is absolute min. and for $Z = 90°$ an absolute max.

In the curves discussed in this treatise heretofore, $q = 90°$ and $Z = 90°$ have been companions very often in giving, alike, numerical max. or min. But here they part company, one giving absolute min., the other giving absolute max.

19. The Computation of the Altitude.

Given t, L, d, *to find* h.

(*a*) For error in h owing to error in t we have the reciprocal to (75) and an interchange of max. and min.

(*b*) For error in h owing to error in L we have the reciprocal to (101) and max. and min. interchanged.

(*c*) For error in h owing to error in d,

$$t, L, \frac{h}{d}; A, c, \frac{a}{b}.$$

Differentiating (69),

$$-\sin a\, da = -\cos c \sin b\, db + \sin c \cos A \cos b\, db; \quad . \quad . \quad . \quad . \quad . \quad (107)$$

$$\frac{da}{db} = \frac{+\cos c \sin b - \sin c \cos b \cos A}{+\sin a} =; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (108)$$

and by (19),

$$= \frac{\sin a \cos C}{\sin a} = \cos C. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

$$\frac{dh}{dd} = \cos q. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (110)$$

Therefore absolute min. when $q = \left\{ \begin{matrix} 90^\circ \\ 270^\circ \end{matrix} \right\}$ and numerical max. when $q = \left\{ \begin{matrix} 0^\circ \\ 180^\circ \end{matrix} \right\}$ from algebraic max. and min., the maximum being equal to unity.

Numerical min. > 0 and < 1 for $Z = 90^\circ$ from algebraic max. and min.

Loci. The curve of elongations and the prime vertical for min. The meridian for max.

EXPLANATION OF THE PLATES.

The numbers shown on the plates correspond to the loci numbered in the text.

Each locus is shown separately for the latitudes of 30°, 45°, and 60°, marked respectively A, B, and C.

Each plate shows the stereographic projection of the sphere on the plane of the horizon; the prime-vertical being the axis of x, and the meridian that of y.

On each plate are drawn diurnal circles, i.e., parallels of declination, marked at their intersections with the meridian, the axis of y, as follows:

d' for a star having $+ d > L$;
d'' for a star having $+ d < L$;
d''' for a star having $- d < L$;
d^{iv} for a star having $- d > L$.

Diurnal circles are shown also for $d = 0$, the equator; and for the parallel $- d = L$, as a right line passing through the nadir at infinity, and parallel to the prime-vertical, the axis of x.

The diurnal circle $+ d = L$ is not drawn, but it may readily be imagined as tangent to X at the zenith, Z.

The astronomical triangles are not constructed, since their lines would encumber and obscure the plates for their more important use, that of showing the curves of max. and min. errors. The triangles may easily be completed, mentally, by imagining the vertical circles, through Z, and the hour-circles, through P, as intersecting on the parallels of declination at the points discussed as giving max. and min. errors—thus with co-L, already drawn, forming the triangle.

Lines above the horizon are drawn full; those below, dotted.

The loci are discussed with regard only to the *theoretically* most favorable and least favorable positions of the body (see Introduction, art. 3); and equal respect will be paid to points below the horizon as is paid to points that would be visible to the observer on the earth.

Again, indetermination may occur at the point that otherwise would be theoretically the best, and the close vicinity to this point will be a favorable region.

Therefore the point that theory declares to be the most favorably situated may be referred to one that is *practicably* the best from the theoretical point of view. The terms *practicably* and *practically* should not be confounded, as the best practical conditions may be left for the observer to select, after presenting those that in theory are the best.

Since true maxima and minima are such with respect to the values of the function that lie immediately on each side of them, neither the several maxima being equal to one another, nor the several minima equal to one another; and since, also, of alternate true maxima and

minima, a particular maximum need not be so great as a minimum that is not alternately near; and since, moreover, disregarding signs, out of several *numerical* maxima of the function, one may be the greatest, and one the least, in numerical value (and the same respecting numerical minima),—therefore we shall avoid, in defining the points in the first instance, unless there can be no mistaking them, the use of the terms *most* favorable and *least* favorable; and shall say *favorable* and *unfavorable*, these terms corresponding respectively to *numerical* min. and *numerical* max., whichever algebraic sign the function may have. The terms *most* and *least* favorable may afterwards be used with discrimination.

In no case of these loci has it been found necessary to have recourse to the second differential of the expression for the error, in order to discriminate the maxima and minima. In No. 1 the method of discriminating is given, for illustration; but it is deemed unnecessary to repeat this proceeding in detail in all cases, since what is obvious to the writer will be obvious to the reader.

A *positive* error in the datum is assumed for the discussion of maxima and minima values in the error of the computed azimuth. Evidently, to make the expressions serve in practice as *corrections*, the proper sign of the error found to have existed in a given part must be used; and, also, the sign of the expression for the error must, itself, be changed.

It will be necessary to keep in mind the method *used in this treatise* in reckoning the angles t, Z, and q, and the explanation of the terms *maximum* and *minimum*, as here used. The reckoning of Z differs from that of astronomers, generally, who reckon from the south point of the horizon. But the writer considers the method used in this treatise the rational one for the purpose in view; and there seemed no way of escape from making terms with the maxima and minima. (See pp. 19–21.)

Though P in the diagrams may be either the north or south pole, whichever is the elevated pole, so long as d of the same name as the latitude is regarded as positive, and if of the contrary name as negative, yet, for convenience and uniformity, the plates are for north latitude. Therefore with respect to the plates we refer to points north and south, as well as east and west.

In referring to circumpolar stars we may have those with $d < L$, possible only when $L > 45°$; one with $d = L = 45°$ as the single case; those whose $d > L$, which may be called *close* circumpolar stars, for they will possess the *general* characteristics of stars very near P, p being small, and they cannot have a $d < 45°$.

There will be stars to consider whose $d > L$, yet not circumpolar stars, possible only when $L < 45°$; and the stars whose $d < L$ cannot be circumpolar ones when $L < 45°$.

In referring to the plates, only a brief of the matter from the text will be given, and abbreviations will be largely used.

PLATES.

LOCUS NO. 1.—ALTITUDE-AZIMUTH ERROR IN h.

[Summary of pp. 25, 26, 31–36, 55–63.] $\frac{dZ}{dh} = \frac{1}{\tan q \cos h}$. . . . . (55)

Branches: 1. Absolute max., the merid. *NPS*.

2. Absolute min., curve of elongations *Pa'Za''* and *e'P'e''*.

3. Num. min., from alg. max. and min., curve *c'Eb'Zb''Wc''*.

It is obvious that the merid. and $q = 90°$ curves give no alg. max. or min.; for the sign of (55) depends on tan q, since cos h is always positive. (55) therefore changes sign when q passes through the values 0°, 90°, 180°, 270°.

At b'', W, and c'', q lies between 0° and 90°; therefore tan q is (+), and alg. max. occur with respect to alg. min. at b', E, and c', where tan q is (—), since $q > 270°$ and $< 360°$. But on a given par. of dec. these alg. max. and min. are equal numerically, and both are num. min. as compared with abs. max. on merid.

Following the star in its diurnal course, beginning at lower cul., we find:

For $+d > L$, at d', abs. max. ∞; a', abs. min. 0 *favorable;* d', abs. max. ∞; a'', abs. min. 0 *favorable.*

For $+d = L$, at low. cul. abs. max.; at Z indeterminate, but akin to a min. The slightest departure from the zenith gives a determinate case, and a very small error, which increases continuously during the progress of the star towards low. cul., where ∞ occurs.

For $+d < L$, at low. cul. d'', abs. max. ∞; b', alg. min., num. min. (—), *favorable;* d'', abs. max. ∞; b'', alg. max., num. min. (+). *favorable.*

For $d = 0$, on merid. abs. max. ∞; at E and W, num. min., (—) and (+) respectively, *favorable.*

For $-d < L$, num. min. at c' (—) and c'' (+), favorable, but practicably in the horizon.

For $-d = L$. In the projection, this is the asymptote to the curve of alg. max. and min., meeting it as a tangent at infinity, the nadir. Practicably, the horizon gives its *favorable* points.

For $-d > L$, abs. min. 0 at e' and e'', *favorable;* practicably in horizon, if star rises.

The unlettered line parallel to $-d = L$ is the asymptote to the lower curve of elongations. In the projection it crosses the merid. at the distance 2 tan L from the origin; and, from the origin, $-d = L$ lies at the distance tan L. The lower asymptote is, on the sphere, a small circle inclined to the equator, and is tangent at the nadir to the p. v. and to the curve of elong.

For all bodies whose $-d > L$, and $d = 0$, the *most favorable practicable* positions are in the horizon. The constant (instrumental and personal) error in h will be eliminated in the mean of the results of observation of a star east and west of the meridian, if taken on the same parallel of altitude.

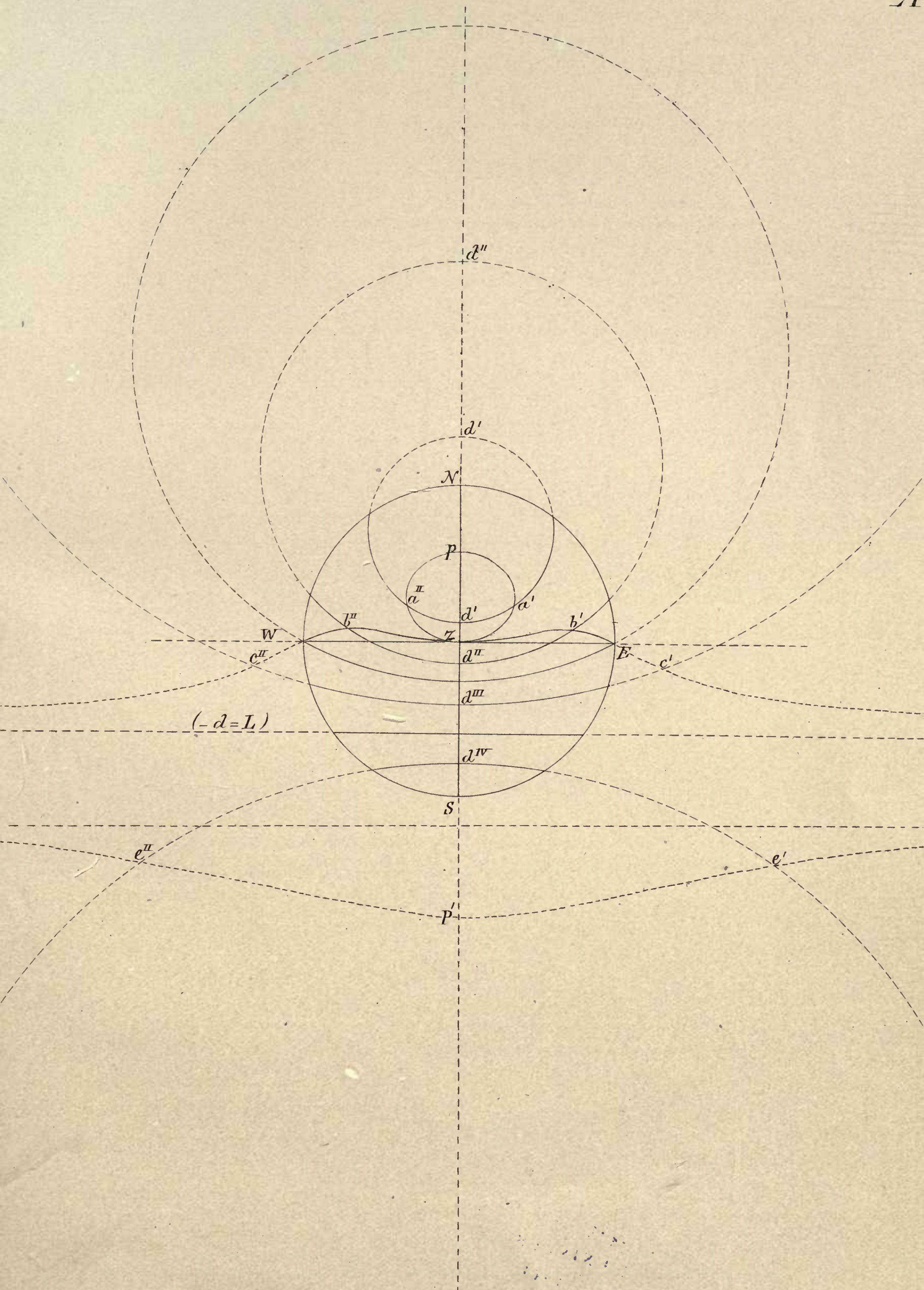
The algebraic max. and min. curve, coinciding with the p. v. in $L = 0°$, swells, with increasing latitudes, until the max. ordinate occurs (or widest departure from the p. v.), in about latitude 54° 31′; it then returns to its first form by the time that latitude 90° is attained.

(Compare with No. 4.)

No. 1

Altitude-Azimuth, error in $\underline{\underline{h}}$

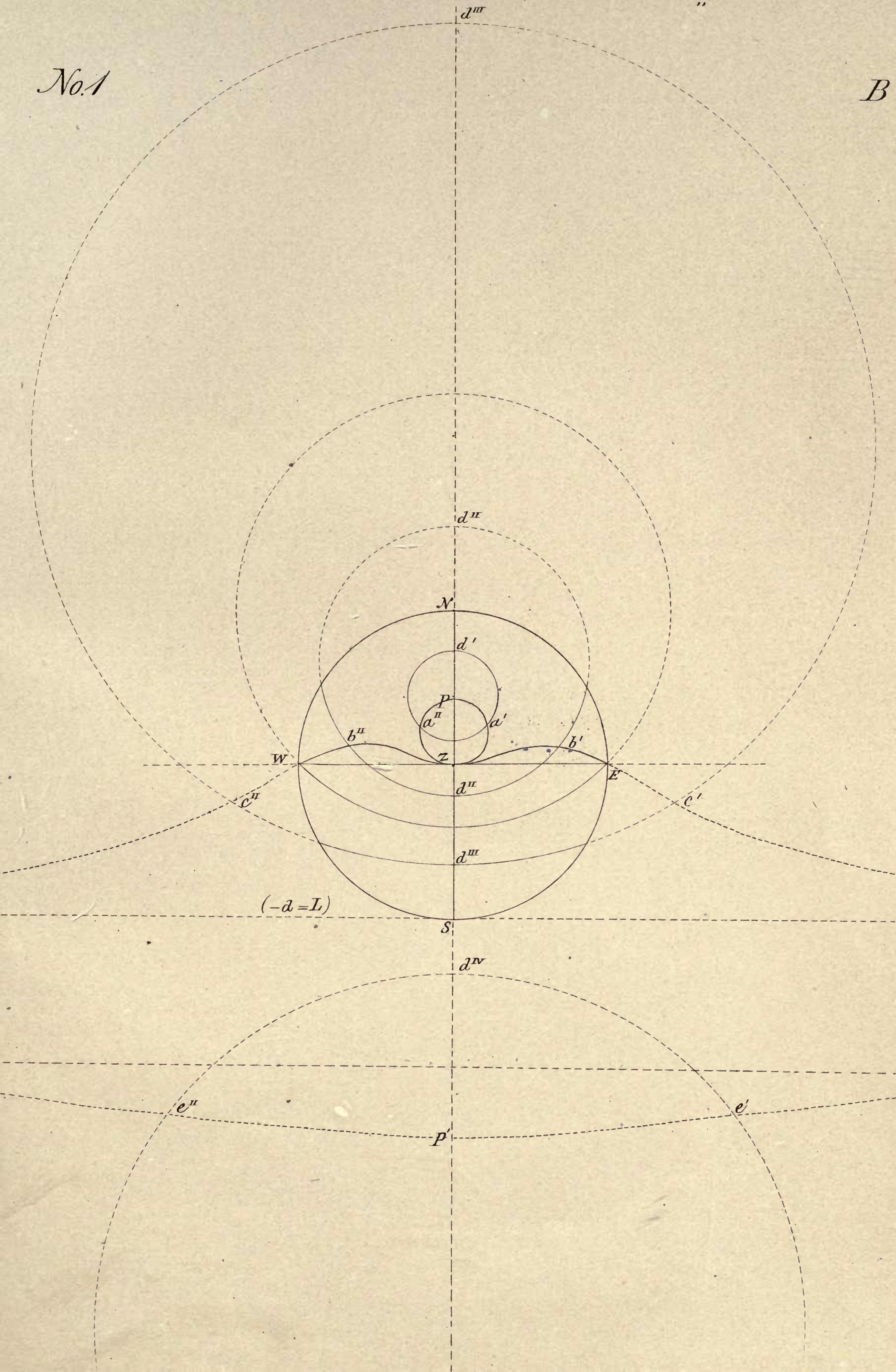
A



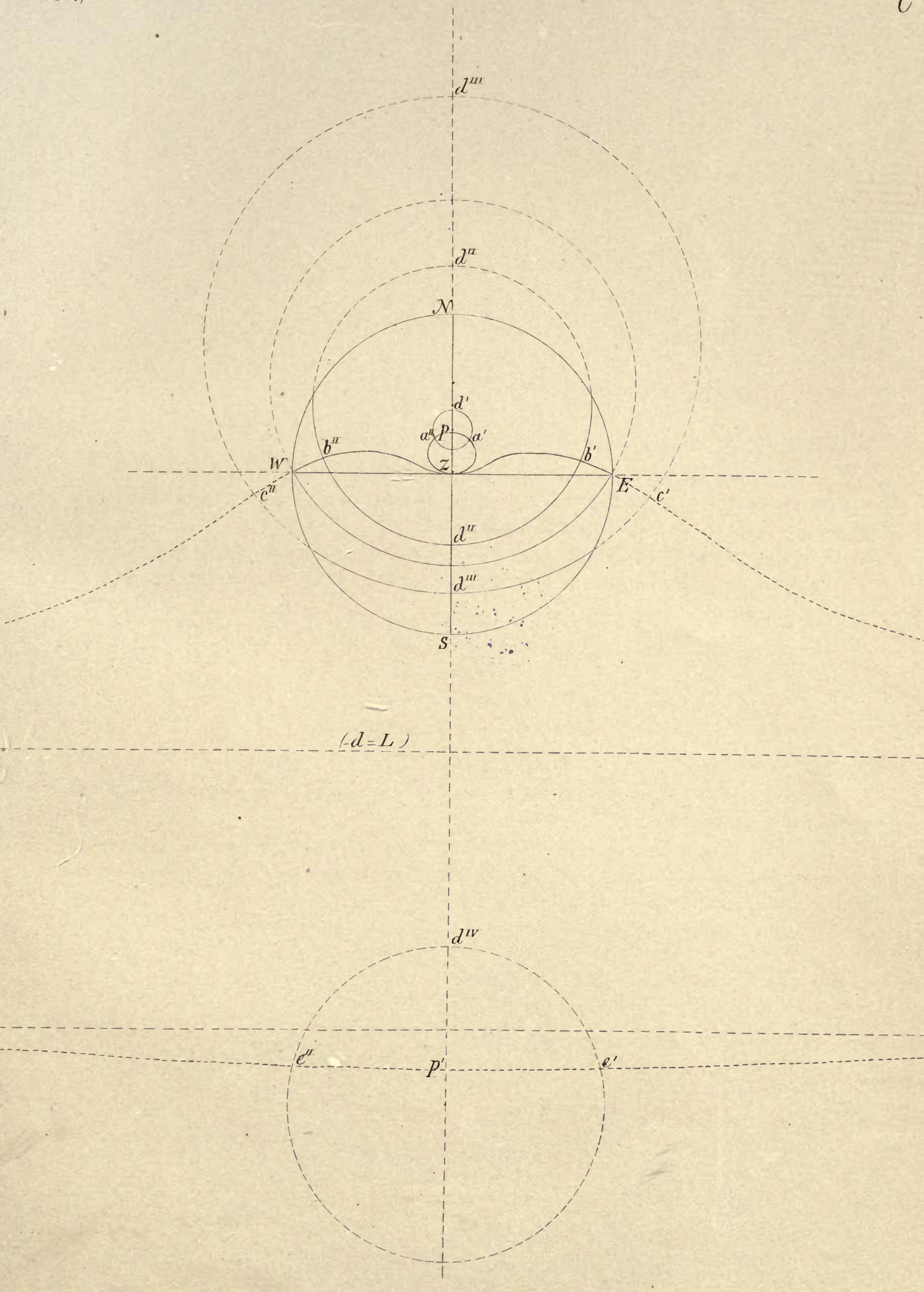
Altitude-Azimuth, error in $\underline{\underline{h}}$

No. 1

B



No. 1, | *Altitude-Azimuth, error in $\underset{''}{h}$* | C



Locus No. 2.—Altitude-Azimuth Error in L.

[Summary of pp. 26, 36, 37, 63, 64.] $$\frac{dZ}{dL} = \frac{1}{\tan t \cos L} \quad \text{. . . . .} \quad (56)$$

Branches: 1. Absolute max., the merid. NPS.

2. Absolute min., the six-hour circle $Pa'b'E \ldots P' \ldots W \ldots P$.

Neither branch gives algebraic max. and min., because tan t changes sign when the given body transits the six-hour circle and the meridian.

The *least favorable* positions for any star are those on the meridian, the error giving ∞.

The *most favorable* positions are on the six-hour circle, giving 0 for error, as follows:

$+d > L$	at	a'	and	a'';
$+d < L$	at	b'	and	b'';
$d = 0$	at	E	and	W;
$-d < L$	at	c'	and	c'';
$-d = L$	at	cutting 6^h circle;		
$-d > L$	at	e'	and	e''.

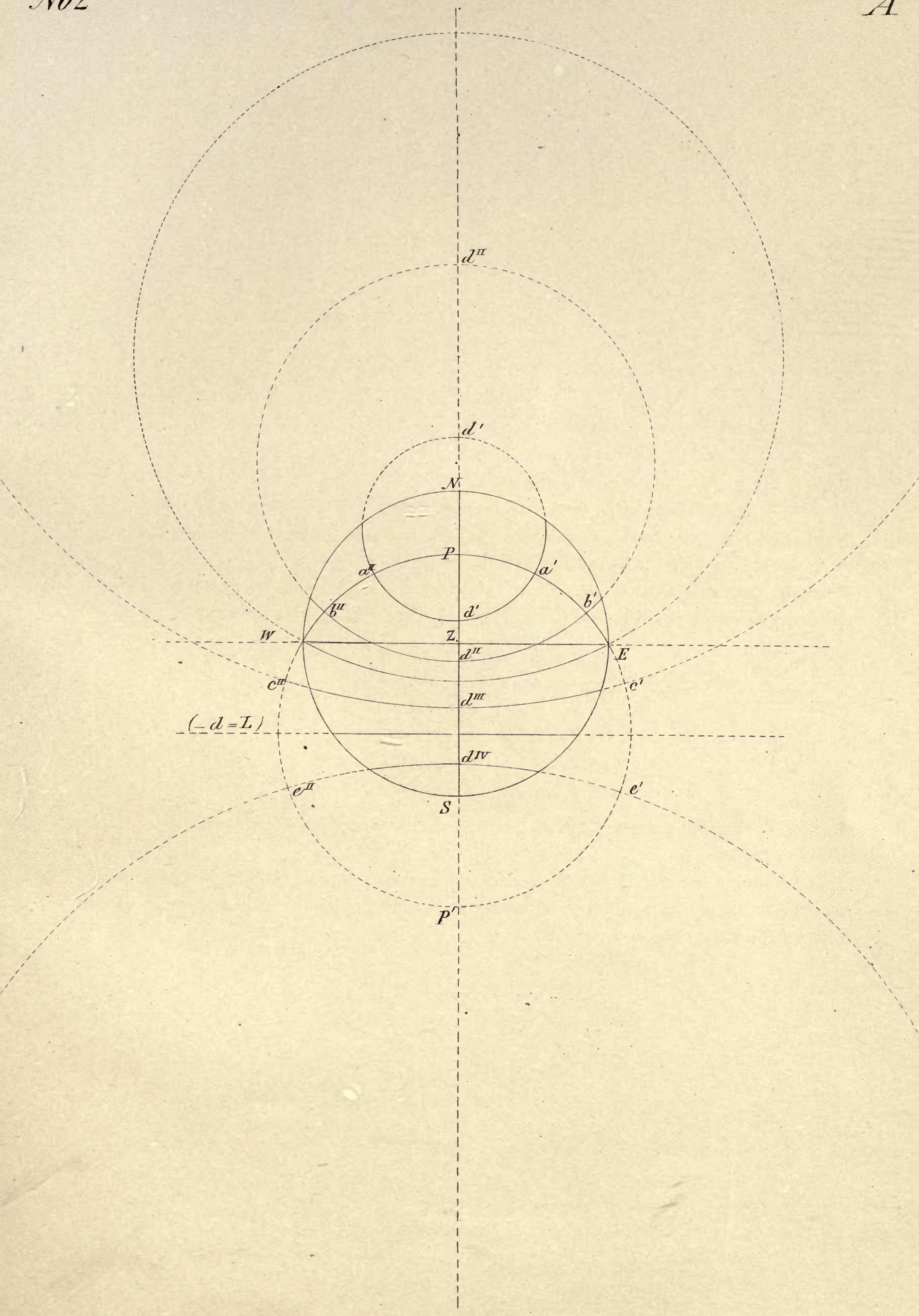
But for all rising and setting bodies having negative declinations the best practicable points are in the horizon.

In determining the direction of the meridian on shore, the error in latitude will be eliminated in the mean of the results of observation of the given body east and west of the meridian on the same parallel of altitude.

Altitude-Azimuth, error in L

No 2

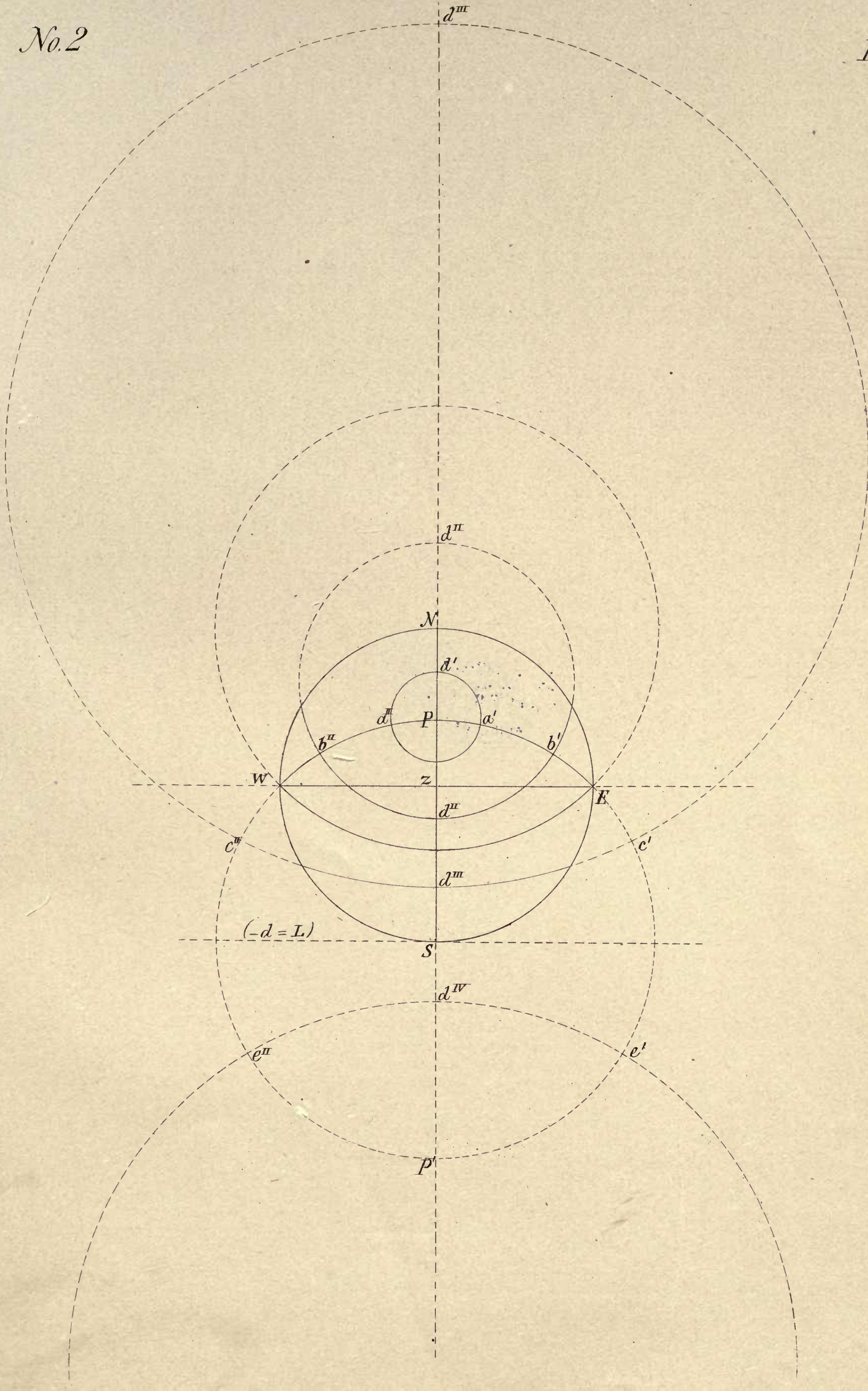
A



Altitude-Azimuth, error in L.

No. 2

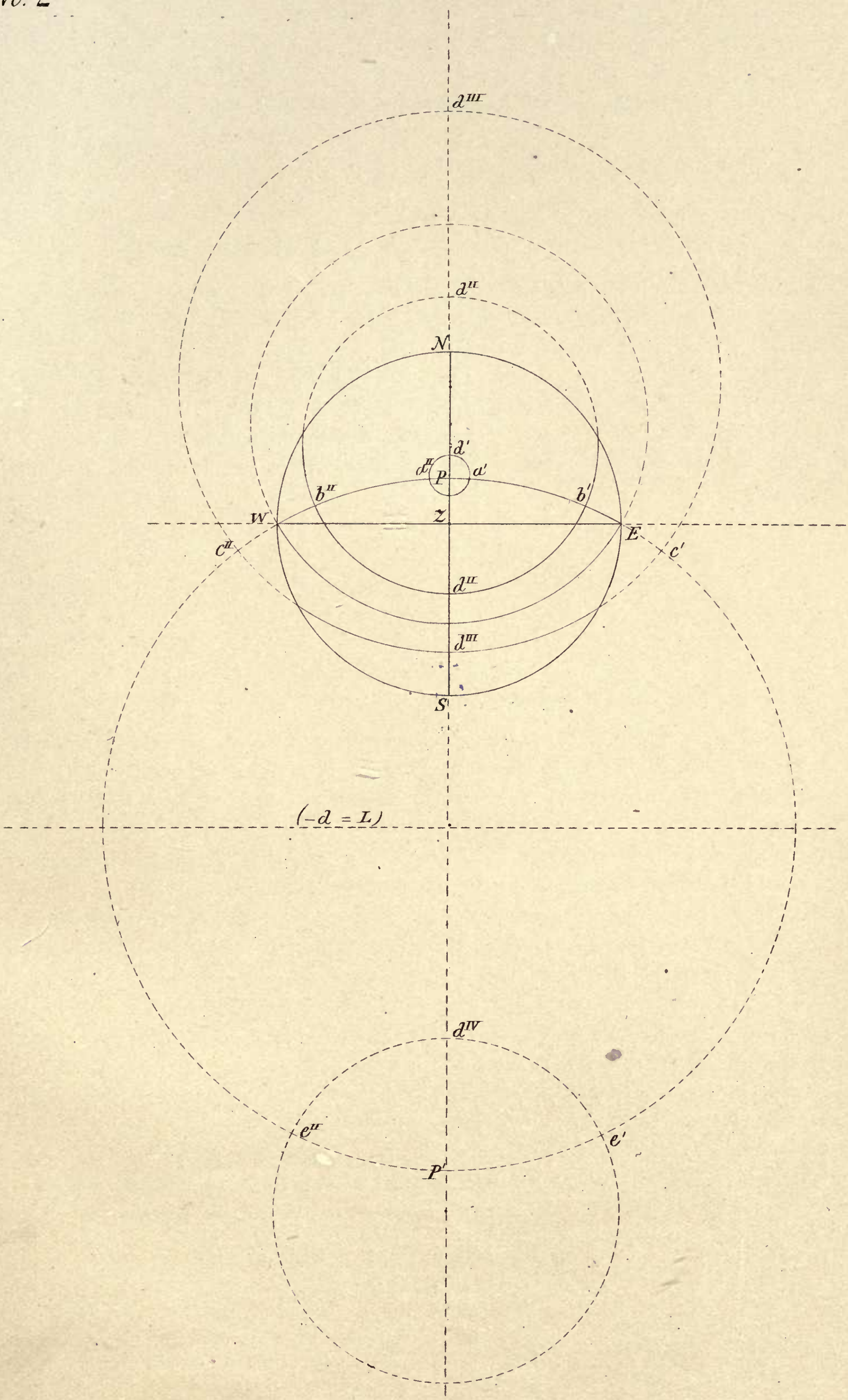
B



No. 2

Altitude-Azimuth, error in L.

C



Locus No. 3.—Altitude-Azimuth Error in d.

[Summary of pp. 26, 37, 64.] $$\frac{dZ}{dd} = -\frac{1}{\sin t \cos L} \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

Branches: 1. Absolute max., the meridian *NPS.*

2. Num. min., from alg. max. and min., the six-hour circle $Pa'b'E \ldots P'$, etc.

These branches, with their kinds of max. and min., are obvious; for sin t changes sign on passing through 0° and 180°; but does not change sign when passing through 90° and 270°.

From the sign of (57) we see that the num. minima, having the same numerical value east and west of the meridian, are alg. min. on the west side and alg. max. when east.

The *least favorable* position for any given body is when on the merid., error $= \infty$.

The *most favorable*, the error having a finite numerical value $=$ sec L, is on the six-hour circle, as follows:

$+d > L$, at a' or a'';
$+d = L$, at points on the 6^h circle;
$+d < L$, at b' or b'';
$d = 0$, at E or W;
$-d < L$, at c' or c'';
$-d = L$, at points on the 6^h circle;
$-d > L$, at e' or e''.

The points for max. and min. are the same as in No. 2, but the num. min. in No. 2 is absolute, 0, for all bodies; while in No. 3 it is equal to sec L for each of all bodies.

For all rising-and-setting bodies having negative declinations, the best practicable points are in the horizon.

If the given body is observed east and west of the merid. on the same par. of altitude, the error will be eliminated in the mean of the results of computing the azimuth; and since this condition is the same regarding error in h and error in L, we should observe as follows:

If the error in L is likely to have a more pernicious effect than the error in h, owing to uncertainty of the geographical position while our instrument and the conditions for observation of the altitude are good,—we should observe the star east and west, on the six-hour circle: then, failing to seize the observation exactly at the points aimed for, the failure to eliminate the error wholly will not have much effect.

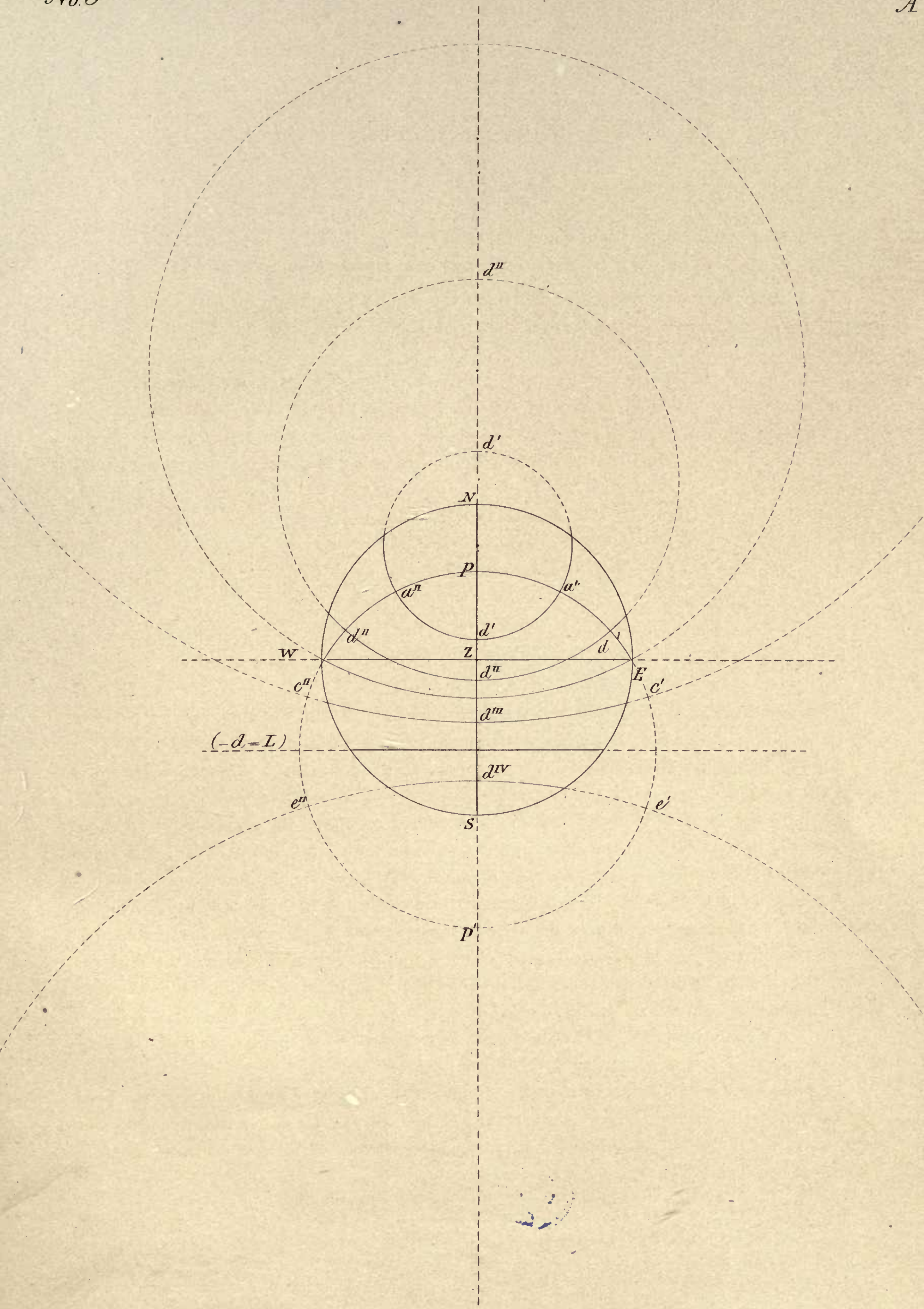
But if the conditions are reversed, and error in h is more to be apprehended, try to seize the body east and west when on the curve of min. error in No. 1.

In moderately high latitudes a close circumpolar star will give $q = 90°$ and $t = 90°$ very near each other, so that whichever is chosen it will be favorable for eliminating the errors in both L and h, and also d. Besides, the altitude will be favorable for the *practical* problem aside from the theoretical.

Altitude-Azimuth, error in $\underset{''}{d}$

No. 3

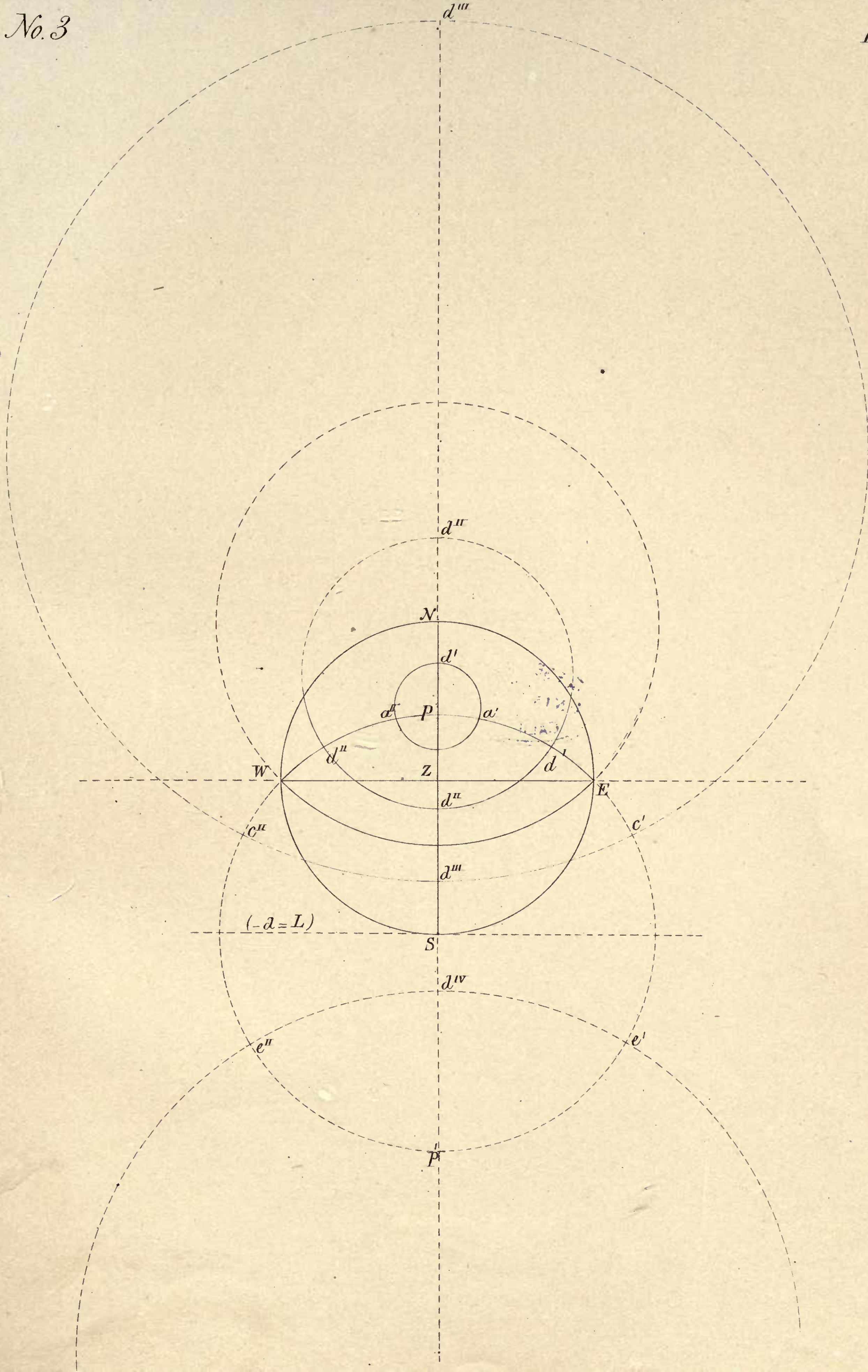
A



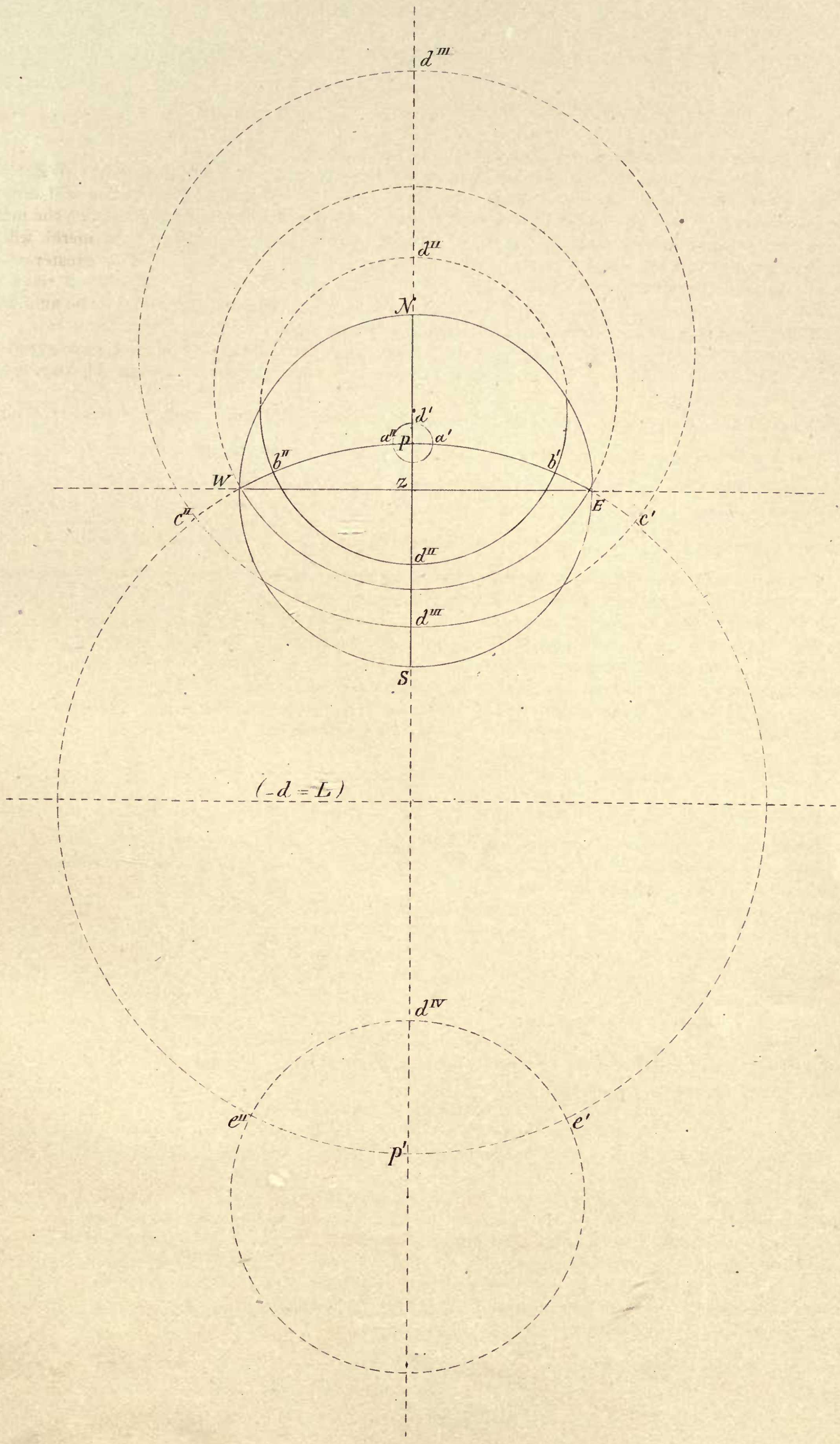
No. 3

Altitude-Azimuth, error in d

B



No 3 Altitude-Azimuth, error in $\underset{\cdot}{d}$ *C*



Locus No. 4.—Time-Azimuth Error in t.

[Summary of pp. 26, 27, 37–41, 64–70.] $\frac{dZ}{dt} = -\frac{\cos q \cos d}{\cos h}$. . . . . (58)

Branches: 1. Absolute min., the curves of elongation $Pa'Za''$ and $e'P'e''$.

2. Alg. max. and min., giving num. max. and min., the merid. NPS and the curve $c'Eb'Zb''Wc''$. The latter gives always numerical *min.*, and pertaining, as it does, to $\pm d < L$, all those stars will cross it unless the latitude is very high. So long as all stars having $\pm d < L$ cross this curve $c'E \ldots c''$, the merid. will give num. *max.* at each culmin. for *all* stars. Because, for $\pm d > L$ the values on the merid. will be numerically greater than the value 0 on the curves of elong.; and for $\pm d < L$ they will be greater on the meridian than on the other curve. But in latitudes $> 70° 32'$ (about) some $\pm d$'s $< L$ will not touch the curve $c'E \ldots c''$; therefore we must look to the other branch of alg. max. and min. for the num. min. This will occur at that culmination of the star that has the less numerical value of h, whether $+$ or $-$. To bring this about, the latitude must be so great that we may say the meridian gives numerical max. *generally*. In (58), since $\cos d$ and $\cos h$ remain positive, the sign of the expression depends on the sign of $\cos q$, dependent in turn on the quadrant in which q is situated.

Taking $+d > L$, alg. max. and min. both occur on the meridian. At upper transit, $q = 180°$, $\cos q$ does not change sign when passing through this value, but is ($-$) on each side; therefore the sign of $\frac{dZ}{dh}$ is ($+$) and we have an alg. max. At lower transit q, having passed through the value 90° (causing $\cos q$ to change sign, and so giving no alg. max. or min., though the value 0 is found) is equal to 0°; and in transit $\cos q$ does not change sign, but is ($+$) on each side, making the sign of $\frac{dZ}{dh}$ ($-$), whence an alg. min. To attain upper culmin. again, q passes through the value 270°, $\cos q$ changes sign, and, though we get 0 for the error, no alg. max. or min. occurs until the max on the merid. recurs.

Therefore for $+d > L$, the num. max. and min., *if governed solely* by algebraic max. and min., would *correspond* to the latter, not only numerically but in the algebraic signs; for the ($+$) error is the greater numerically since $\cos h$ at the upper transit is less than at the lower. But at a' and a'' absolute min., 0, intervene and the alg. max. and min. on the merid. both become num. max.

For $-d > L$ *similar* conditions prevail, but not exactly *like* conditions, for the alg. max. and min. are reversed, both in numerical value and sign; for $\cos h$ is then the greater at upper transit, and q there passes through 0°, and at lower transit through 180°.

Taking $\pm d < L$, q on one side of the meridian remains always between 0° and 90°, and on the other between 360° and 270°. Therefore the sign of $\cos q$ is always ($+$) and the sign of $\frac{dZ}{dh}$ always ($-$). Therefore alg. max. and min. occur alternately; the minima on the meridian (unequal numerically) and the maxima on the other curve (equal numerically). But *our numerical* maxima and minima (unfavorable and favorable points) interchange with the algebraic minima and maxima.

To follow the star in its diurnal course, beginning at the lower culmination:

For $+d > L$. At d', alg. min., num. max. ($-$), *unfavorable*; at a', abs. min. $= 0$, *favorable*; d', alg. max., num. max. ($+$), *unfavorable*; a'', abs. min. $= 0$, *favorable*.—For $+d < L$. At d'', alg. min., num. max. ($-$), *unfavorable*; b', alg. max., num. min. ($-$), *favorable*; d'', alg. min., num. max. ($-$), *unfavorable*; b'', alg. max., num. min. ($-$), *favorable*.—For $+d = L$. Practicably, *favorable* very near Z, the zenith, and *unfavorable* at low. culm. At Z the value is indeterminate.—For $d = 0$. At E and W *favorable*, on the merid. *unfavorable*.—For $-d < L$. At d''', low. culm., alg. min., num. max. ($-$), *unfavorable*; at c', alg. max., num. min. ($-$), *favorable*; at d''', alg. min., num. max. ($-$), *unfavorable*; at c'', alg. max., num. min. ($-$), *favorable*.—For $-d > L$. At low. culm., alg. max., num. max. ($+$), *unfavorable*; at e', abs. min. $= 0$, *favorable*; at d^{IV}, above horizon, alg. min., num. max., *unfavorable*; at e'', abs. min. $= 0$, *favorable*.

For $d = 0$ and for negative d's the horizon furnishes the best practicable points if the body rises and sets.

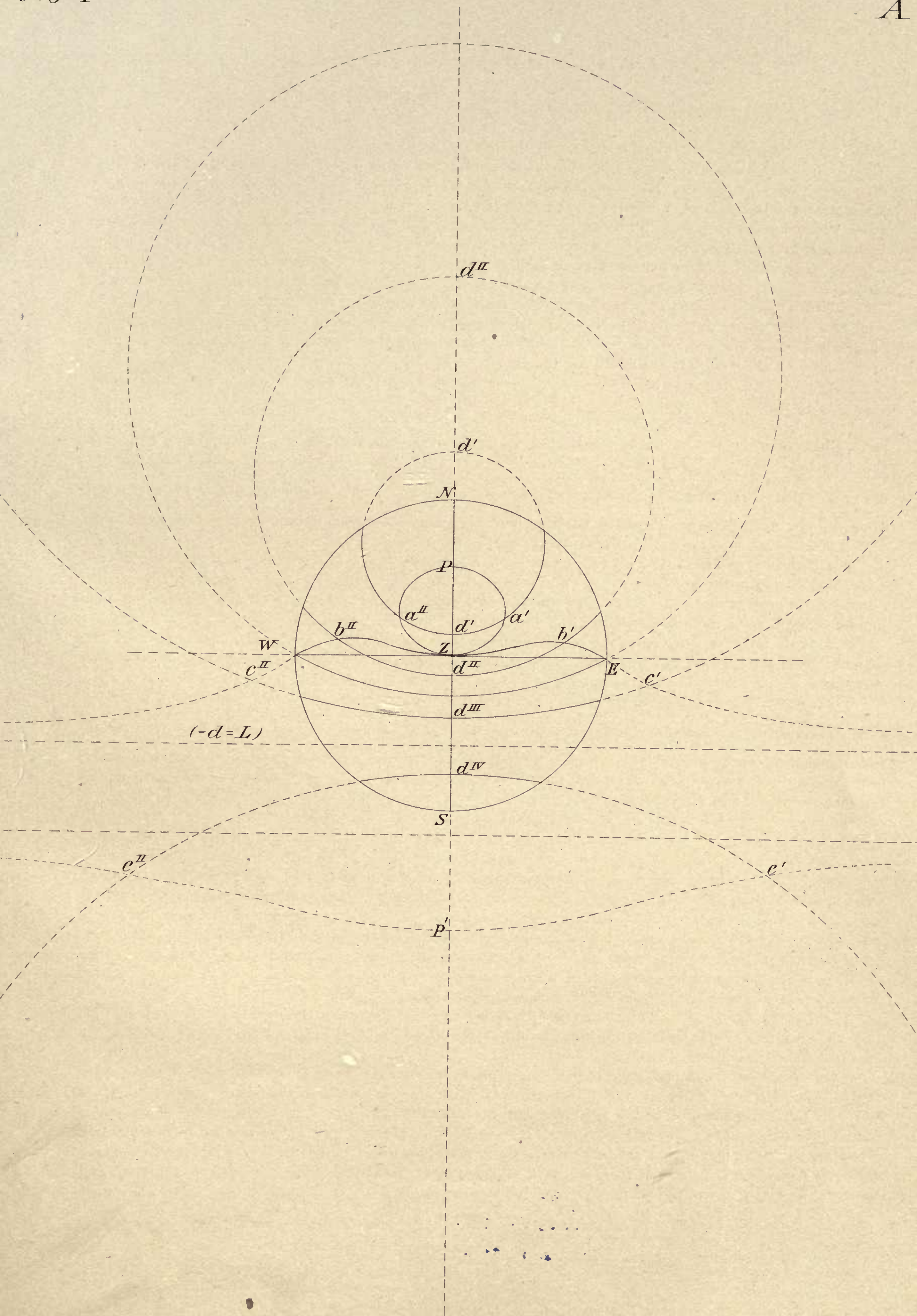
The error in Z due to error in t will not be eliminated by observing the star east and west, symmetrically situated with respect to the meridian.

The asymptotes are the same as those in No. 1; one at the distance from Z equal to $\tan L$, and the other at the distance $2 \tan L$. Both No. 1 and No. 2 have the *spherical ellipse* for giving points of absolute min.; and the other branches look alike in low latitudes; but, both starting with the p. v. in latitude 0°, No. 1 swells out until reaching its limit of growth, when it returns to coincide with the prime-vertical in latitude 90°. No. 2, on the other hand, continues its growth until the branches meet on the merid., then separate, and maintain separation in very high latitudes, until in lat. 90° one branch becomes the horizon (equator), and the other a point at the zenith (pole).

No 4

Time-Altitude-Azimuth, error in t

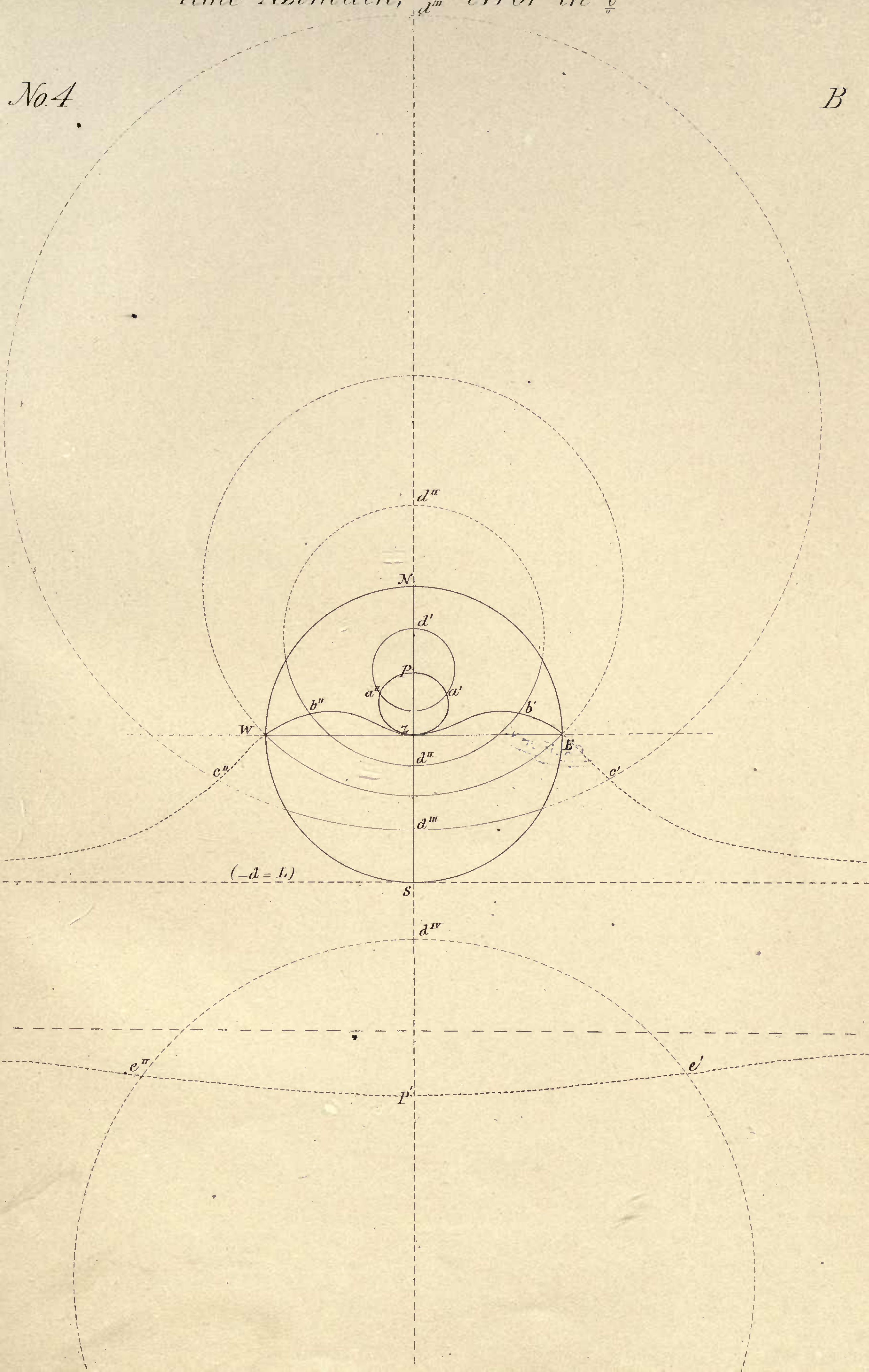
A



Time Azimuth, error in $\underset{''}{t}$

No. 4

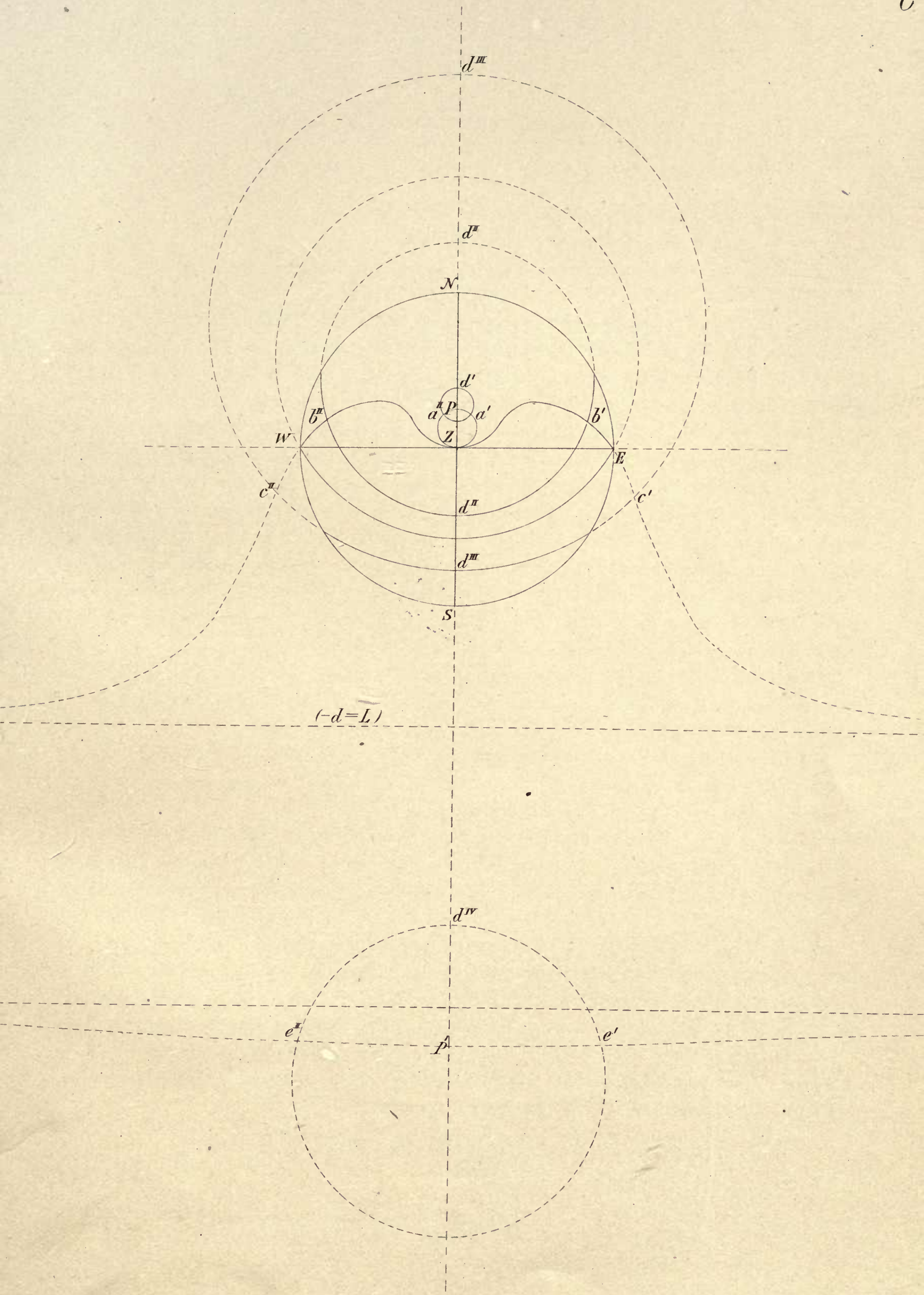
B



No. 4

Time-Altitude-Azimuth, error in t

C



Locus No. 5.—Time-Azimuth Error in L.

[Summary of pp. 27, 41, 42, 70–73.] $\frac{dZ}{dL} = \tan h \sin Z = \text{etc.}$ . . . . (59)

Branches: 1. Absolute min., the horizon *NESW*.

2. Absolute min., the meridian *NPS*.

3. Algebraic max. and min., giving numerical max., the curve

$$Pa'''Zb^{iv}c^{iv}e^{iv}Se'''c'''b'''Za^{iv}P,$$

together with the infinite branches $b'a'Na^{vi}b^{vi}$ and $e'P'e^{vi}$.

Governed by the signs of sin Z and tan h, the alg. max. and min. are alternates. Leaving out of consideration the abs. min. on the horizon and on the meridian, we find for $+d > L$, when not a circumpolar star (Plate A) at a', alg. max.; at a''', alg. min.; at a^{iv}, alg. max.; at a^{vi}, alg. min. But as compared to the absolute min. (now considered), all these alg. maxima and minima are numerical maxima, and give the *unfavorable* points for observation; the *favorable* points being d', a'', d', and a^{v}. Of these, the *more favorable* are a'' and a^{v}, for, notwithstanding the error is zero at each of the four points, in case of failure to seize any one of them exactly, proximity to the horizon is better than to the merid. (And likewise better *practically*, for the lower the altitude the better, so long as the uncertain refraction does not vitiate the condition; and in the case of time-azimuth, the altitude of the star and the errors attending its correct measurement do not enter the problem.) Therefore we may say that the horizon gives the *most favorable* points for the observation.

The good and the bad points are so easily perceived on the plates that it is deemed unnecessary to follow stars other than the one for $+d > L$, already used. But taking in the absolute min. to complete the discrimination in max. and min. of the various kinds, we have:

At d' (lower transit), abs. min. $= 0$; at a', alg. max., num. max. $(+)$; at a'', abs. min. $= 0$; at a''', alg. min., num. max. $(-)$; at d', abs. min. $= 0$; at a^{iv}, alg. max., num. max. $(+)$; at a^{v}. abs. min. $= 0$; at a^{vi}, alg. min., num. max. $(-)$.

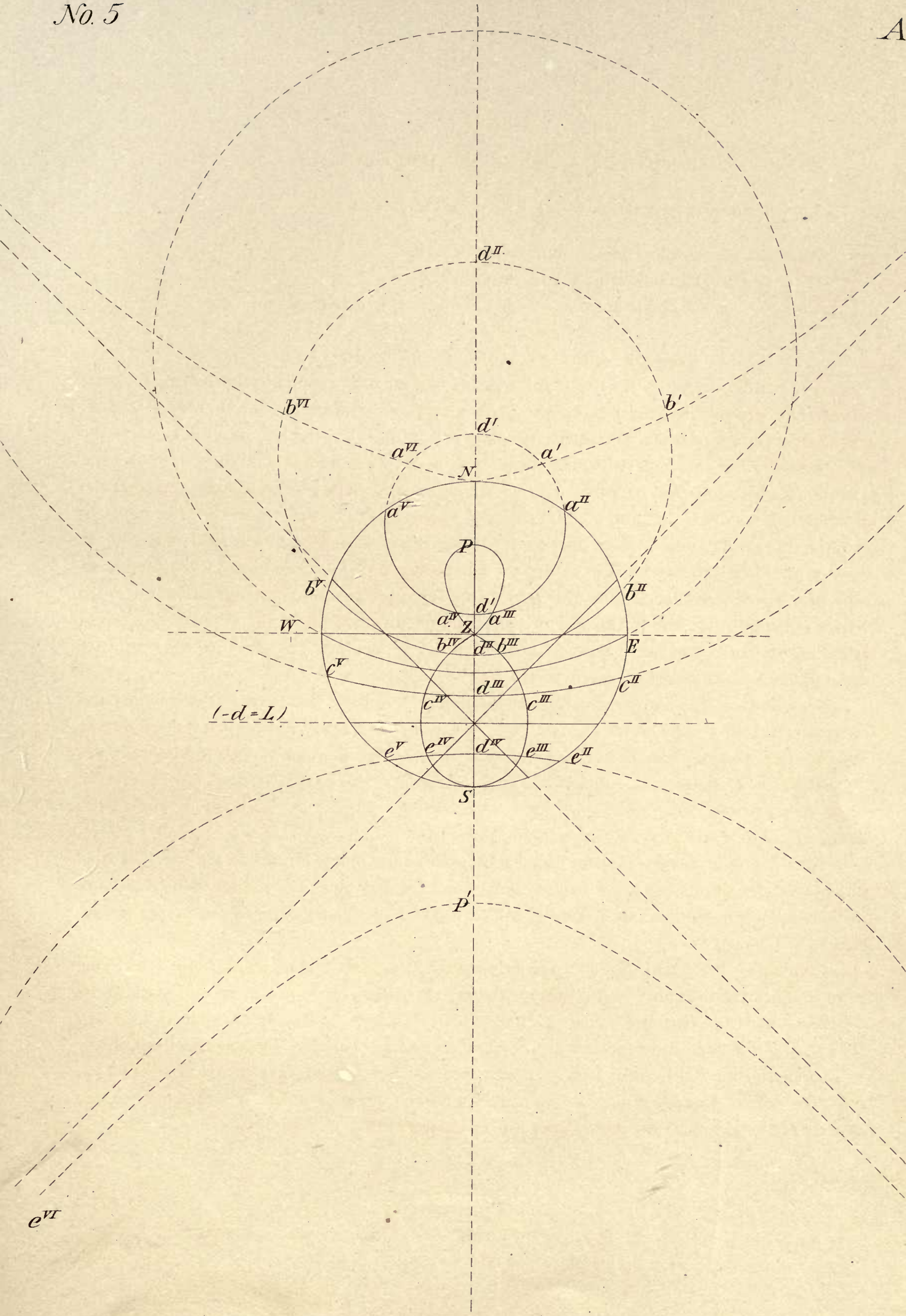
Taking $+d > L$ when the star is a circumpolar one (Plates B and C), we see that it is most favorably situated at lower culmination; since for a given error in azimuth in the seizure of the star on the meridian, tan h sin Z will be less than its value close to the upper culmination.

Comparing No. 4 with No. 5, it will be seen that the most favorable position considering error in t is near the most unfavorable with respect to error in L, in the case of stars whose $d > L$, even if not circumpolar; if a circumpolar star, these antagonistic points approach closer; and if a close circumpolar star, they are very near together. But the error in latitude will be eliminated if the star is observed both east and west at the same relative points to the meridian. Therefore select $q = 90°$ and $270°$ to reduce the error in t to 0 if the star is seized exactly at those points, and nearly zero if near those points.

Time-Azimuth, error in L.

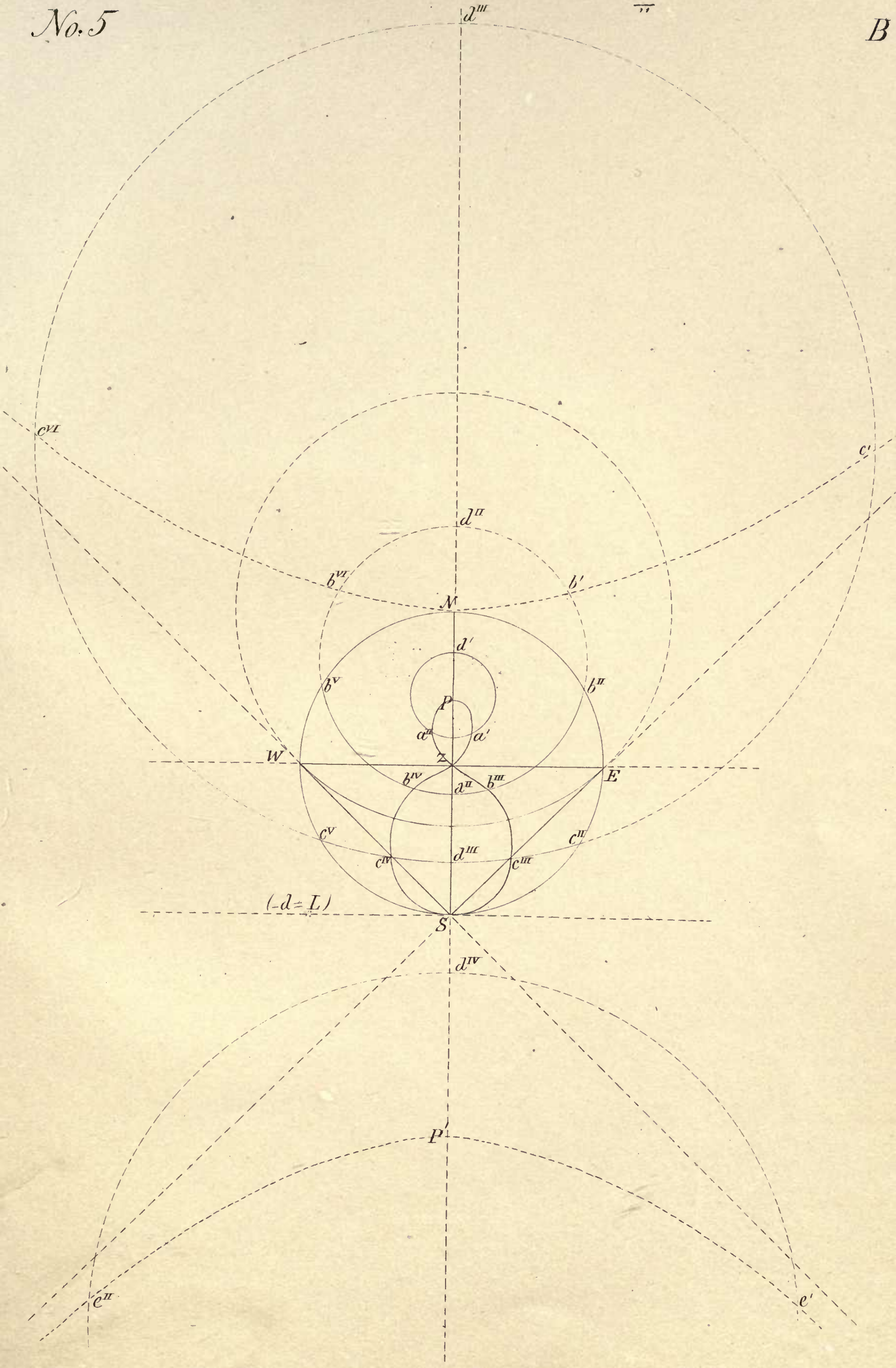
No. 5

A



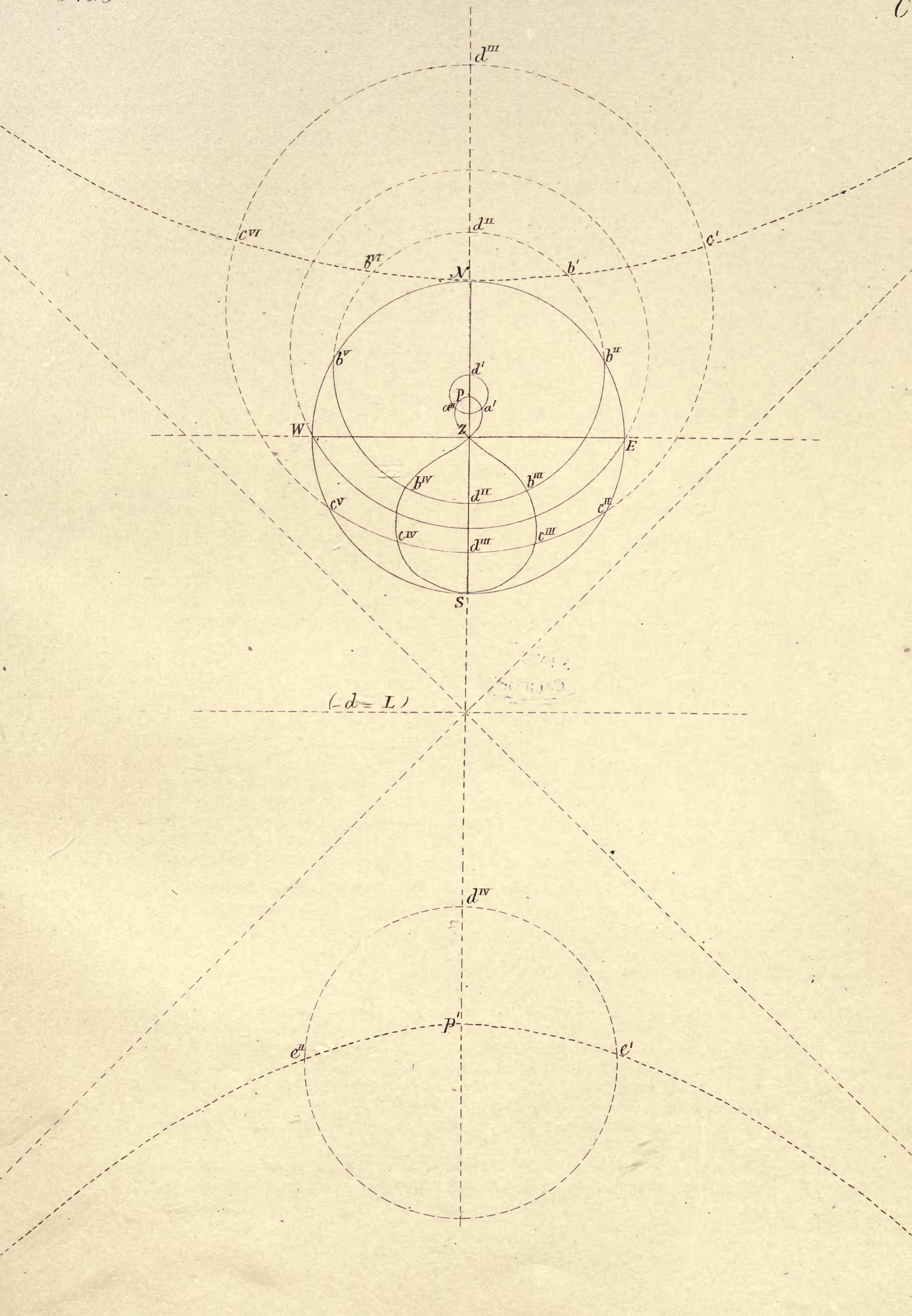
Time-Azimuth, error in L.

No. 5 B



No. 5

Time-Azimuth, error in L.



Locus No. 6.—Time-Azimuth Error in d.

[Summary of pp. 27, 42–44, 73–76.] $\frac{dZ}{dd} = -\frac{\sin q}{\cos h}$. . . . . (60)

[Note.—The investigation of Locus No. 6 (time-azimuth error in d) by its equations (186) and (187), or inspection of the corresponding diagrams, indicates that for some latitudes and declinations there may be three points on either side of the meridian where the diurnal path of a body may cross the locus. These three points are determined by the three roots of the cubic equations referred to above.

By reference to the diagram for Locus No. 6 (A), it is readily seen that for latitude 30° some bodies cross the locus three times on each side of the meridian, while others cross it but once, and it is clear that for some particular declination the diurnal circle will be *tangent* to the locus.

The relation between a given latitude, L, and the declination, d, causing the diurnal circle to be tangent to the locus, is found by determining the condition that will give equal roots to cos t and sin h, in their respective equations.

The equation (186) is the more convenient for this purpose. Writing A for tan L, and B for tan d, the equation becomes

$$\cos^3 t + (A^2 + B^2 - 1)\cos t - 2AB = 0. \quad \text{(a)}$$

Should either A or B be greater than unity, the coefficient of cos t will be positive; and by the theory of equations two roots will be imaginary. Thus at the outset we are limited to the consideration of latitudes and declinations not exceeding 45°.

The condition for equal roots in (a) is

$$\left.\begin{aligned} &(A^2 + B^2 - 1)^3 + 27A^2B^2 = 0; \\ \text{or}\quad &A^2 + B^2 - 1 = -3\sqrt[3]{A^2B^2}. \end{aligned}\right\} \quad \text{(b)}$$

Expanding (b) and arranging, we have

$$B^6 + (3A^2 - 1)B^4 + (3A^4 + 21A^2 + 1)B^2 + (A^2 - 1)^3 = 0.$$

This is a cubic equation in B^2; and if A (tan L) is known, we can solve for B^2 by Horner's method of approximation, and thence deduce B. The two values of B thus found, equal numerically, with opposite signs, determine the two declinations (North and South), giving tangency to the locus.

Having found B, we may easily deduce the value of cos t corresponding to the point of tangency; for, substituting from (b) $-3\sqrt[3]{A^2B^2}$ for $A^2 + B^2 - 1$ in (a), we have

$$\cos^3 t - 3\sqrt[3]{A^2B^2}\cos t - 2AB = 0;$$

the roots of which are always $\quad -\sqrt[3]{AB}, \quad -\sqrt[3]{AB}, \quad$ and $\quad 2\sqrt[3]{AB}$. . . . . (c)

Therefore $\cos t = -\sqrt[3]{AB}$ gives the value of t corresponding to the point of tangency; and from (c) we may observe that, for the point of *cutting* the locus, cos t is opposite in sign and numerically twice as great as for the point of tangency.

A particular case of interest is that in which the declination causing tangency is equal to the latitude.

Put $\quad B = A$ in (b).

We have $\quad 8A^6 + 15A^4 + 6A^2 - 1 = 0;$ . . . . (d)

whence $\quad (8A^2 - 1)(A^2 + 1)(A^2 + 1) = 0.$

The equation in A^2 has one positive root, $A^2 = \frac{1}{8}$, and two negative roots, $A^2 = -1$. The two last give imaginary values for A. From the first we obtain $A = \pm \frac{1}{4}\sqrt{2}$. Log $A = 9.5484550$. Whence $L = d = 19° 28' 16''.39$, giving tangency to the locus.]

Branches: 1. Abs. min., the merid. *NPS*.

2. Alg. max. and min., giving num. max. and min., the curve $c'Eb'Za''Pa'Zb''Wc''$, and the infinite branch $e'P'e''$.

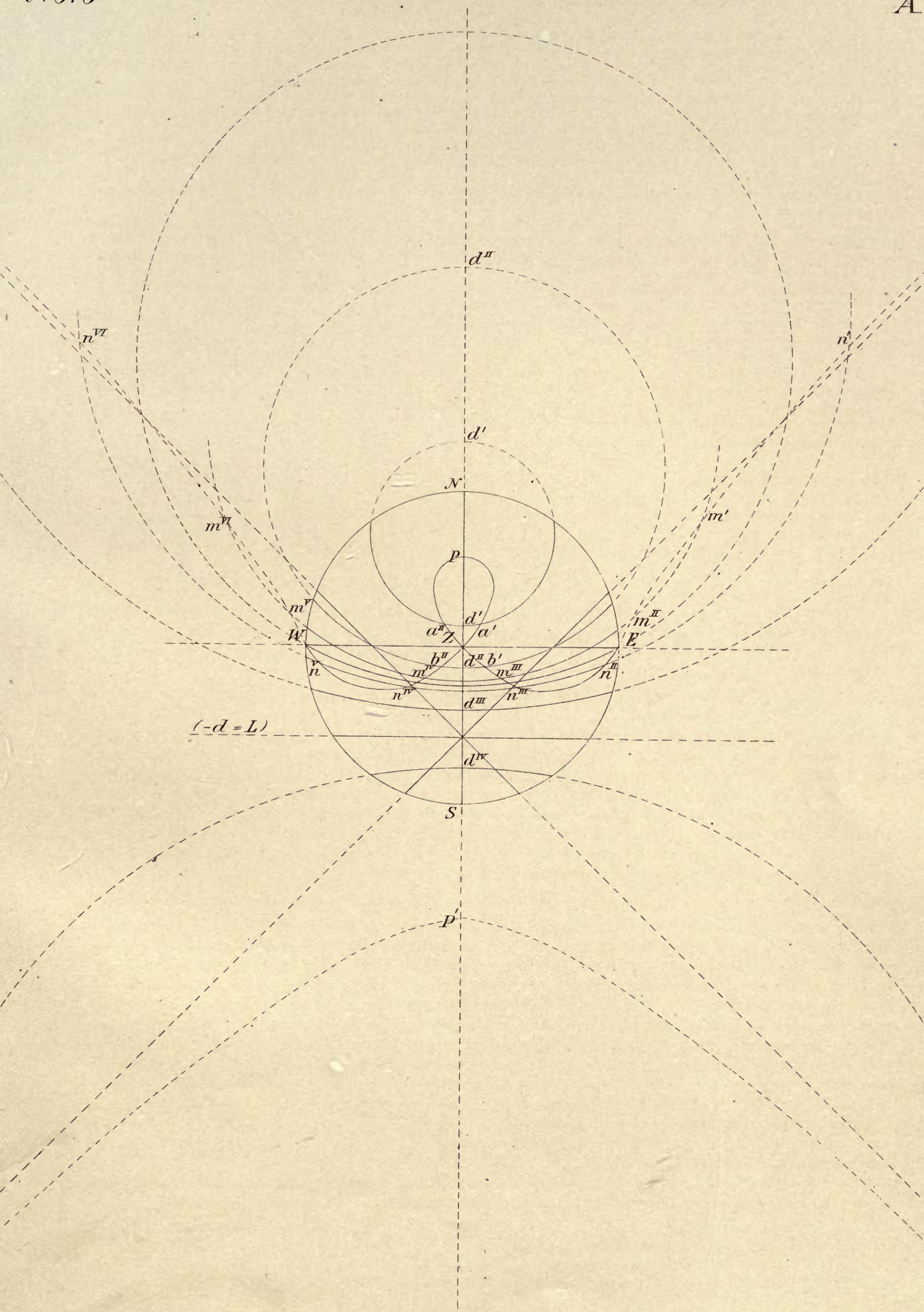
In lats. 45° to 90° every par. of dec. will cut the curve of alg. max. and min. (on either side of the merid.) once, and only once. In lats. < 45° each par. of dec. cuts at least once; some one north par. in the given lat. intersects once and at another point is tangent; so also some one south par. of dec.: some parallels cut the curve three times. Therefore Plate A presents more features than do B and C. Looking at one side of the merid. only, if $L > 45°$, the curve meets its asymptote at infinity only, the nadir; if $L = 45°$, the asymptote is tangent to the curve at E and then meets it again only at infinity; if $L < 45°$, the asymptote crosses the curve once above the horizon, once below, and then meets it at infinity. Alg. max. and min. points for parallels of dec. that intersect the curve once on each side of the merid. may easily be discriminated, since they depend on the sign of sin q alone. In case the parallel cuts thrice, the num. value of $\frac{\sin q}{\cos h}$ must be considered in connection with its sign. But, since the points are of alternate max. and min., no difficulty will be experienced in distinguishing the max. from the min., though from inspection of (60) the relative numerical values are not obvious. Recollecting that sin q is (+) west of the merid. and (−) east, let us follow the star characterized by m', m'', etc.: at low. cul., abs. min. 0; at m', alg. max., num. max. (+); at m'', alg. min., num. min. (+); at m''', alg. max., num. max. (+); at upper cul., abs. min. 0; at m^{iv}, alg. min., num. max. (−); at m^{v}, alg. max., num. min. (−); at m^{vi}, alg. min., num. max. (−). The tangent parallels of dec. are not drawn. At their points of cutting the curve will occur num. max., and at tangency no max. or min., but a decreasing or increasing function as may be, *algebraically*. For $+d > L$, max. error at a' and a'': if the star is not circumpol., practicably the most favorable position is exactly on merid. at upper cul., but in horizon is good; if a circumpol. star, then at low. cul. For $+d = L$ the choice is lower cul.; at Z the error is indeterminate, but close to Z very large. For $+d < L$, if a circumpol. star, at low. cul. best; but if not circumpolar, at upper cul., though the horizon may be good; the max. lying at b' and b''. For all stars with (−) dec. observe on merid.

Comparing No. 4 with No. 6, the remarks on No. 5 apply here, by substituting for *error in L* the words *error in d;* and considering all the errors in the data, the conclusions are the same as given in No. 5.

Time-Azimuth, error in d

No. 6

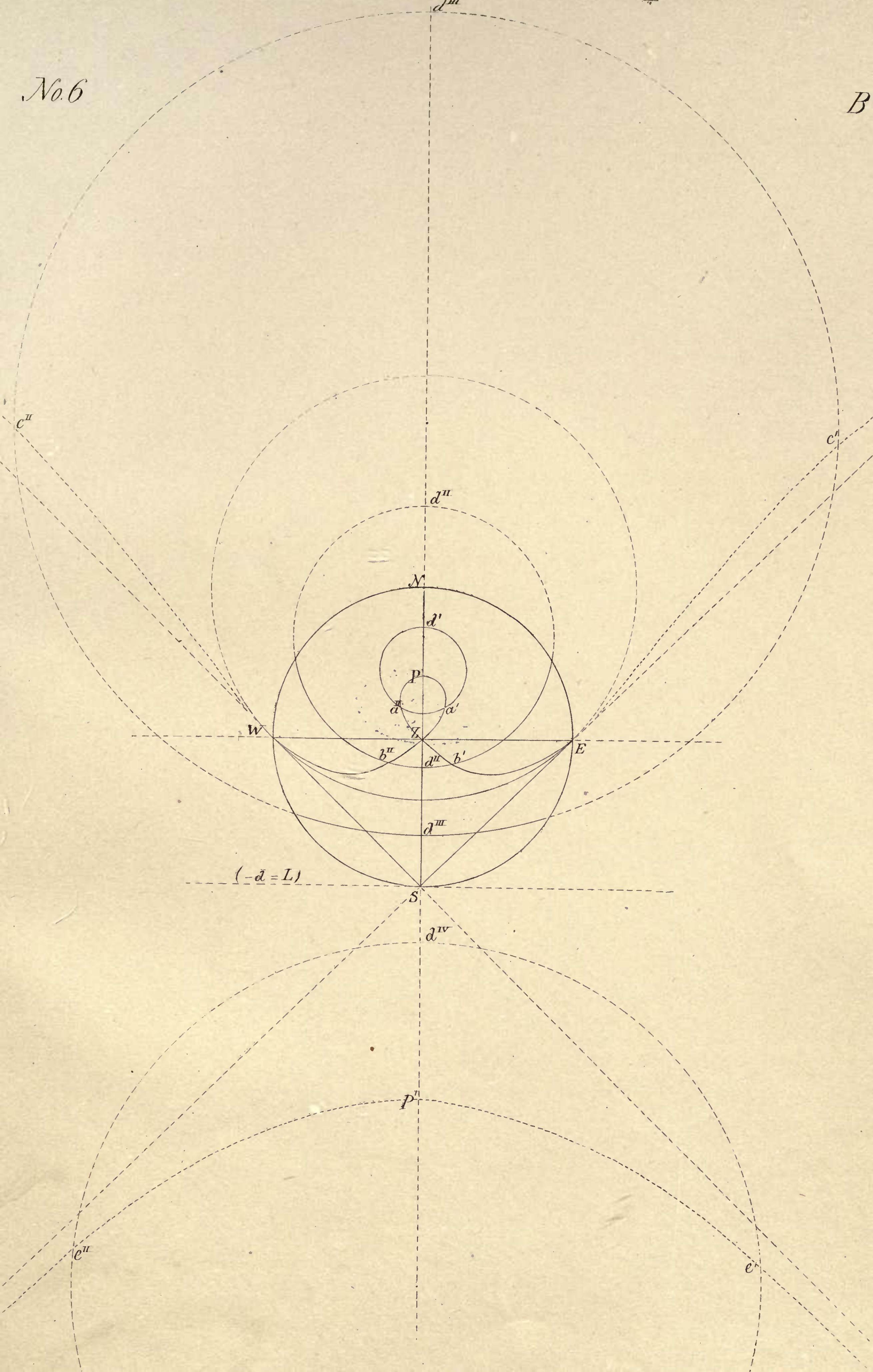
A



Time-Azimuth, error in d

No. 6

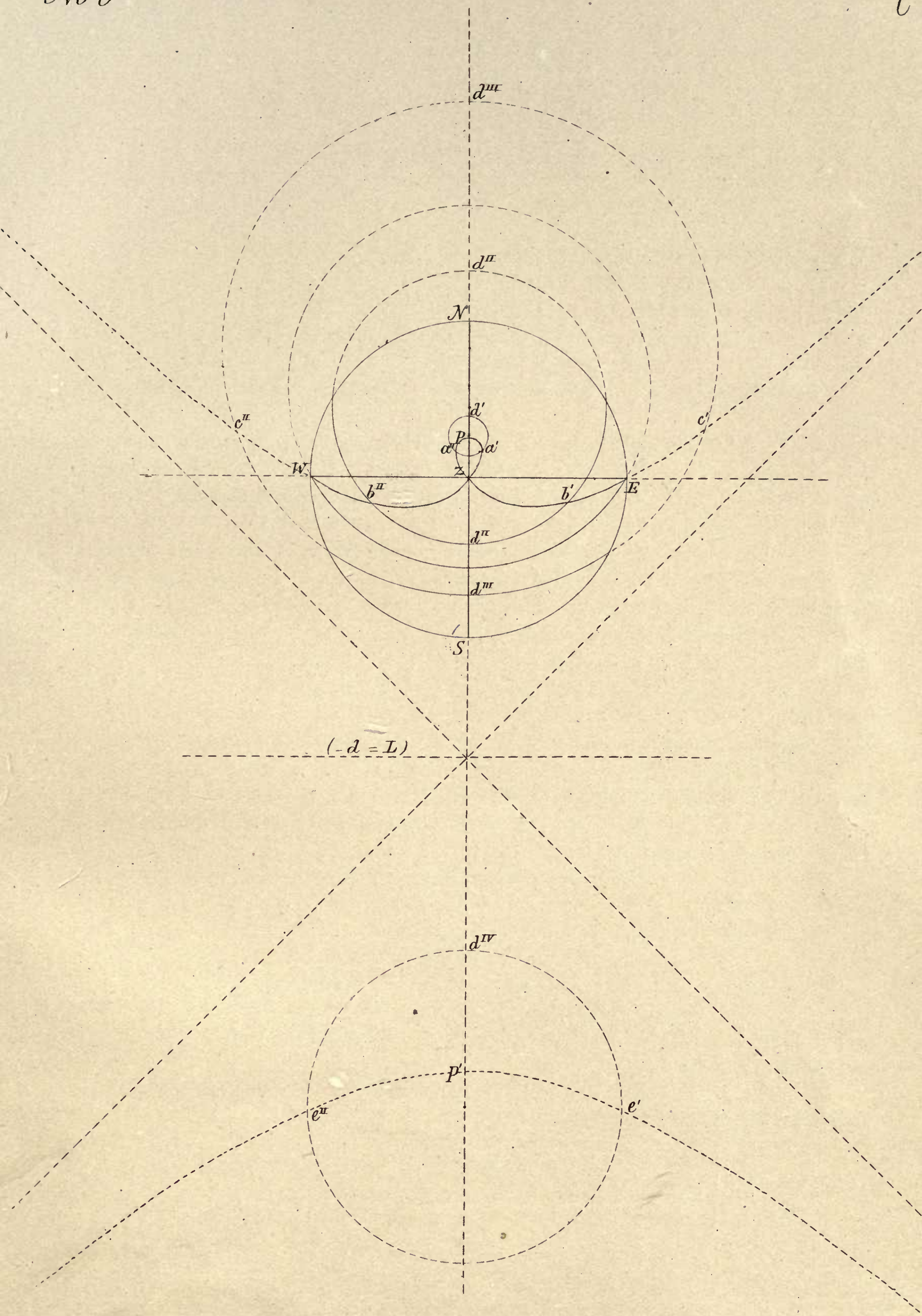
B



Time-Azimuth, error in $\underset{''}{d}$

No 6

C



Locus No. 7.—Time-Altitude-Azimuth Error in *h*.

[Summary of pp. 27, 28, 44–46, 76–80.] $\frac{dZ}{dh} = \tan Z \tan h$. . . . . (61)

Branches: 1. Absolute max., the prime-vertical *WZE*.
2. Absolute min., the merid. *NPS*.
3. Absolute min., the horizon *NESW*.
4. Alg. max. and min., giving numerical max., $Pa'''Za^{iv}$ and $Na'b'Ec'''e'''S$ . . . . W . . . . $b^{vi}N$ and $e'P'e^{vi}$.

Considering only the curve of alg. max. and min., it is evident that on either side of the merid.:

If $+d > L$ and circumpolar, the star will cross the curve once, above the horizon only; if not circumpolar, once below and once above the horizon. If $+d = L > 45°$, one contact only, in the zenith; if $+d = L = 45°$, a point of contact in Z and one in N; if $+d = L < 45°$, contact at Z and a cut below horizon. If $+d < L$, and circumpolar star, no points of contact; if not circumpolar, one cut below horizon. If $d = 0$, the equator at E cuts the curve. If $-d < L$, one point of intersection, above horizon. If $-d > L$, once above and once below the horizon, if a rising and setting body; if not, there is one point, below only.

Now, including the branches of absolute max. and min., and attending to the signs of tan Z and tan h, we have:

For $+d > L$ (not circumpolar, Plate A), at d', low. cul., abs. min. 0; at a', alg. max., num. max. (+); at a'', abs. min. 0; at a''', alg. min., num. max. (—); at d', abs. min. 0; at a^{iv}, alg. max., num. max. (+); at a^{v}, abs. min. 0; at a^{vi}, alg. min., num. max. (—).

For the close circumpol. star $d > L$ (Plates B and C), at low. cul. d', abs. min. 0; a', alg. min., num. max. (—); d', upper cul., abs. min. 0; a'', alg. max., num. max. (+).

For $+d < L$, at low. cul. d'', abs. min. 0; b', alg. max., num. max. (+); b'', abs. min. 0; b''', abs. max. ∞; at upper cul., abs. min. 0, etc.

For $d = 0$, at E and W indeterminate, and in their vicinity bad conditions. Observe at transit on the merid.

For $-d < L$, at low. cul. abs. min. 0; c', abs. max. ∞; c'', abs. min. 0; c''', alg. max., num. max. (+); d''', upper cul., abs. min. 0; c^{iv}, alg. min., num. max. (—), etc.

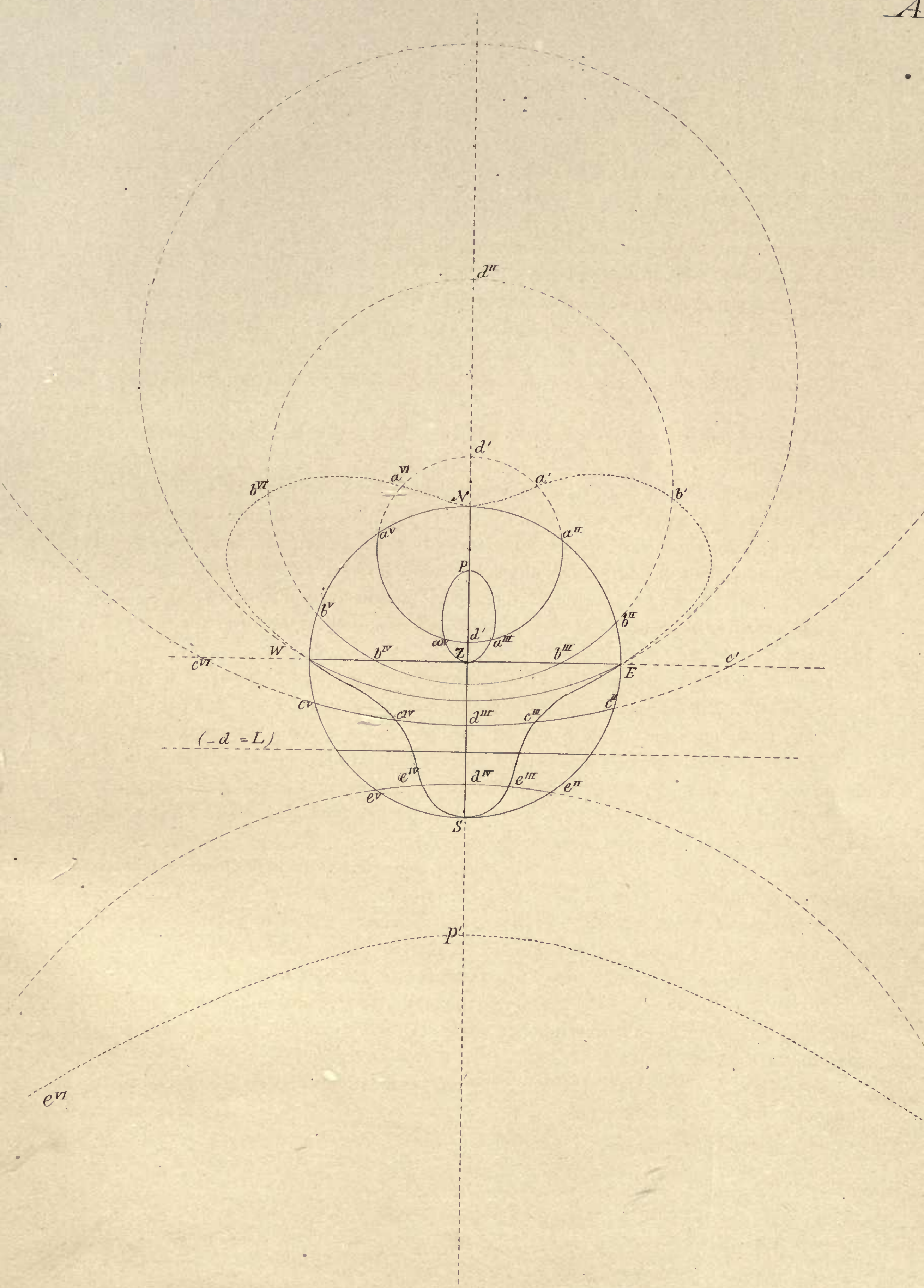
For $-d > L$, e'', d^{iv}, and e^{v}, abs. min. 0; while at e''' and e^{iv}, num. max.

In any case, the choice of the horizon or the merid. to give abs. min. points depends on their relative proximity to a point of max. error. Observations east and west may be made to eliminate the error.

For $+d = L$, avoid the zenith, as giving indetermination; and its vicinity as giving a large error, and observe at low. cul.

No 7

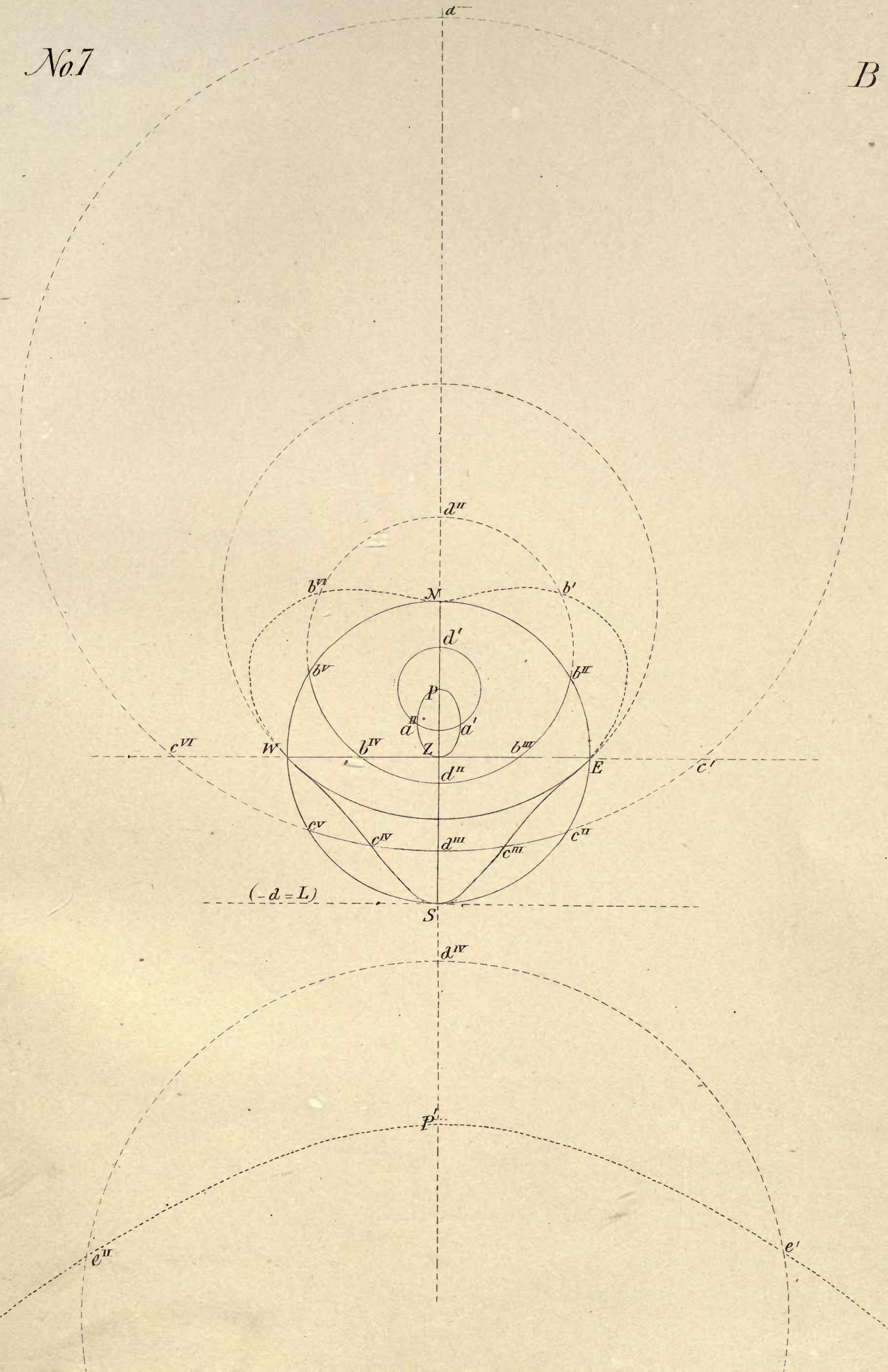
Time-Altitude-Azimuth, error in $\underline{h}$



Time-Altitude-Azimuth, error in $\underline{h}$

No.7

B



LOCUS No. 8.—TIME-ALTITUDE-AZIMUTH ERROR IN t.

[Summary of pp. 28, 46–48, 80, 81.] $\frac{dZ}{dt} = \frac{\tan Z}{\tan t} = \frac{\cos d \cos t}{\cos h \cos Z}$. . . . . (62)

Branches: 1. Abs. max., the p. v. WZE.
2. Abs. min., the six-hour circle $Pb'EP'$.
3. Alg. max. and min., giving num. max. and min., the merid. NPS.

Not considering the abs. max. and min., the alg. max. and min. occur as follows:

For $+d > L$, at low. cul. $\cos t$ is $(-)$ and $\cos Z (+)$; $\therefore dZ$ is $(-)$ and at upper cul. $\cos t$ and $\cos Z$ both $(+)$; therefore an alg. min. at lower and alg. max. at upper cul. whatever the relative values of $\cos h$. But these values make the numerical max. and min. correspond to the algebraic. For $d = L$ the same, but at upper cul., regarding Z as changing from $< 90°$ to $> 270°$, not passing into the 2d or 3d quadrant, the absolute max. will be also an *algebraic* max. $= +\infty$. Therefore at low. cul., alg. min., and at upper cul., alg. max. correspond to num. min. and max.—For $+d < L$, at both culminations the function is $(-)$; hence the value of $\cos h$ governs. For d between $d = L$ and $d = 0°$,

$$\text{at low. cul.} \left\{ \begin{array}{l} h \text{ is less numerically} \\ \cos h \text{ is greater numerically} \\ dZ \text{ is less numerically} \end{array} \right\} \text{than at upper cul.}$$

Therefore at low. cul. we find an algebraic max., and at upper cul. an algebraic min., but numerically the names are changed. For $d = 0$ the error has a constant numerical value and is $(-)$ at each culmination; hence there is neither max. nor min. of any kind. For $-d < L$ at both culminations the function is $(-)$;

$$\text{at low. cul.} \left\{ \begin{array}{l} h \text{ is greater numerically} \\ \cos h \text{ is less numerically} \\ dZ \text{ is greater numerically} \end{array} \right\} \text{than at upper cul.}$$

Therefore an alg. min. at low. cul. and alg. max. at upper cul.; but numerically the names are changed.

For $-d > L$, alg. max. at low. cul., num. max. $(+)$; and alg. min. at upper cul., num. min. $(-)$.

Considering, now, the absolute max. and min. in connection with the preceding, we have for *numerical* max. and min. on the meridian the same name as given by the algebraic max. and min., except for $d > \pm L$. For, *these* stars not crossing the p. v., at both transits occur numerical max.

For $+d > L$, at low. cul. alg. min., num. max. $(-)$; a', abs. min. 0; at a' (upper), alg. max., num. max. $(+)$; a'', abs. min. 0. For $d = L$, at low. cul. alg. min., num. max. $(-)$; on six-hour circle, abs. min. 0; at Z, alg. max. $+\infty$, absolute max. ∞; on six-hour circle, abs. min. 0. For $+d < L$, at low. cul., alg. max., num. max. $(-)$; b', abs. min. 0; b'', absolute max. ∞; at upper cul. alg. min., *num. min.* $(-)$ (*greater*, however, *numerically* than the *numerical max.* $(-)$ at low. cul.: in *most* cases our numerical max. is numerically greater than our numerical min., distinguished from alg. max. and min. by disregarding algebraic signs; but the intervening absolute max. and min. now interchange the names of numerical max. and min., as first assigned when comparing with only alg. max. and min.); at b''', absolute max., ∞; b^{iv}, absolute min. 0.—For $d = 0°$, a constant value to dZ, equal to $-\frac{1}{\sin L}$.—For $-d < L$, at low. cul., d''', alg. min., num. min. $(-)$; c', abs. max. ∞; c'', abs. min. 0; d''', upper cul., alg. max., *num. max.* $(-)$ (*less numerically* than the *num. min.* at lower cul.); at c''', abs. min. 0; c^{iv}, abs. max. ∞.—For $-d = L$, at low. cul., alg. max., $+\infty$ (Z changing from $> 90°$ to $< 270°$), absolute max. ∞; on six-hour circle, abs. min. 0; at upper cul., alg. min., num. min. $(-)$; on six-hour circle, abs. min. 0.—For $-d > L$, at low. cul., alg. max., num. max. $+$; e', abs. min. 0; upper cul., alg. min., num. max. $(-)$; at e'', abs. min. 0.

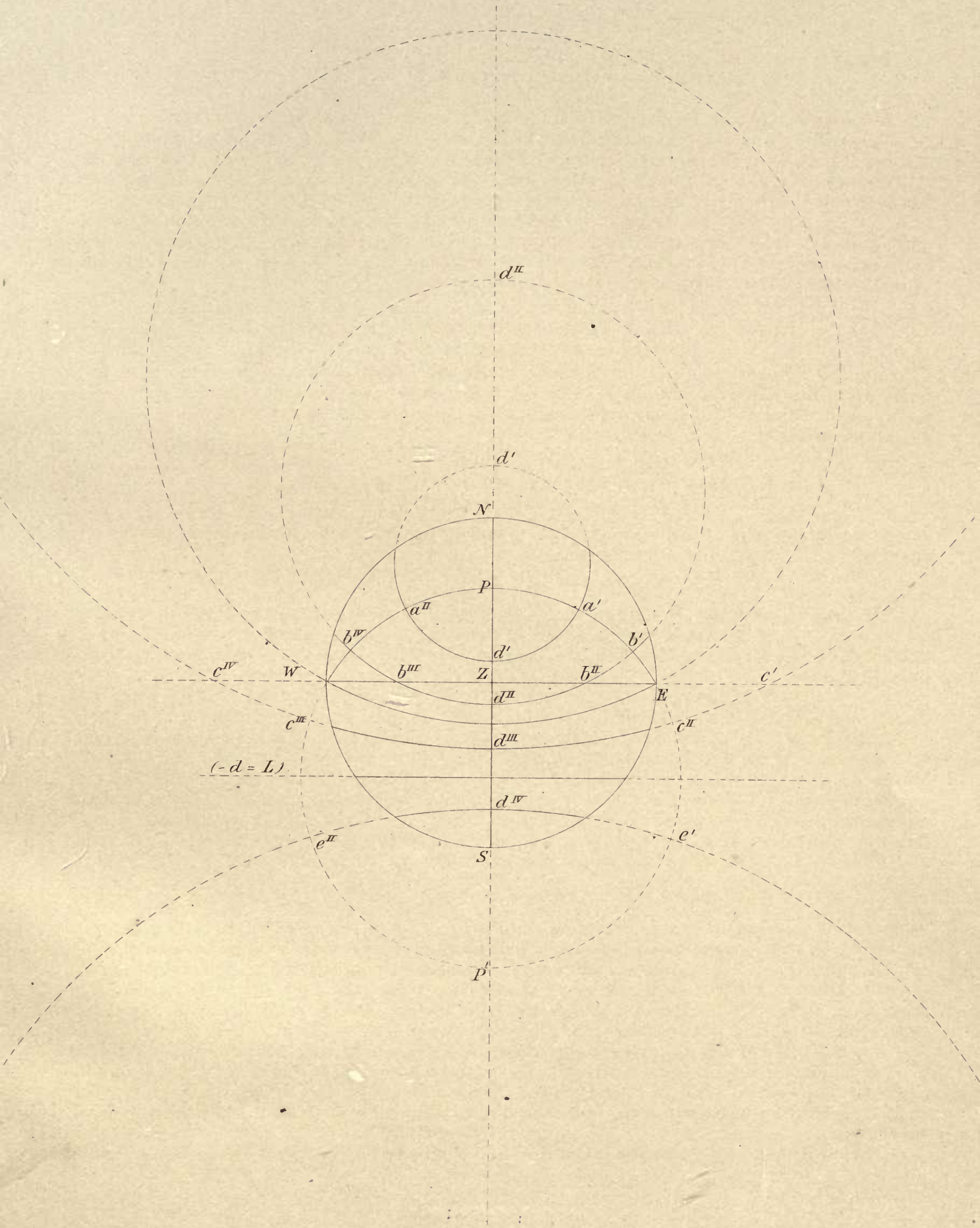
For all $+d$'s avoid observing on that side of the six-hour circle *towards* the p. v. For all $-d$'s the horizon is theoretically the best place to seize the star at. Practically, both these conditions must yield something to the best conditions for the altitude (a given part of the triangle entering the problem), and, therefore, a point near the meridian, in bearing, may be chosen.

The error in Z owing to error in t will not be eliminated by observations east and west.

No 8

Time-Altitude-Azimuth, error in t

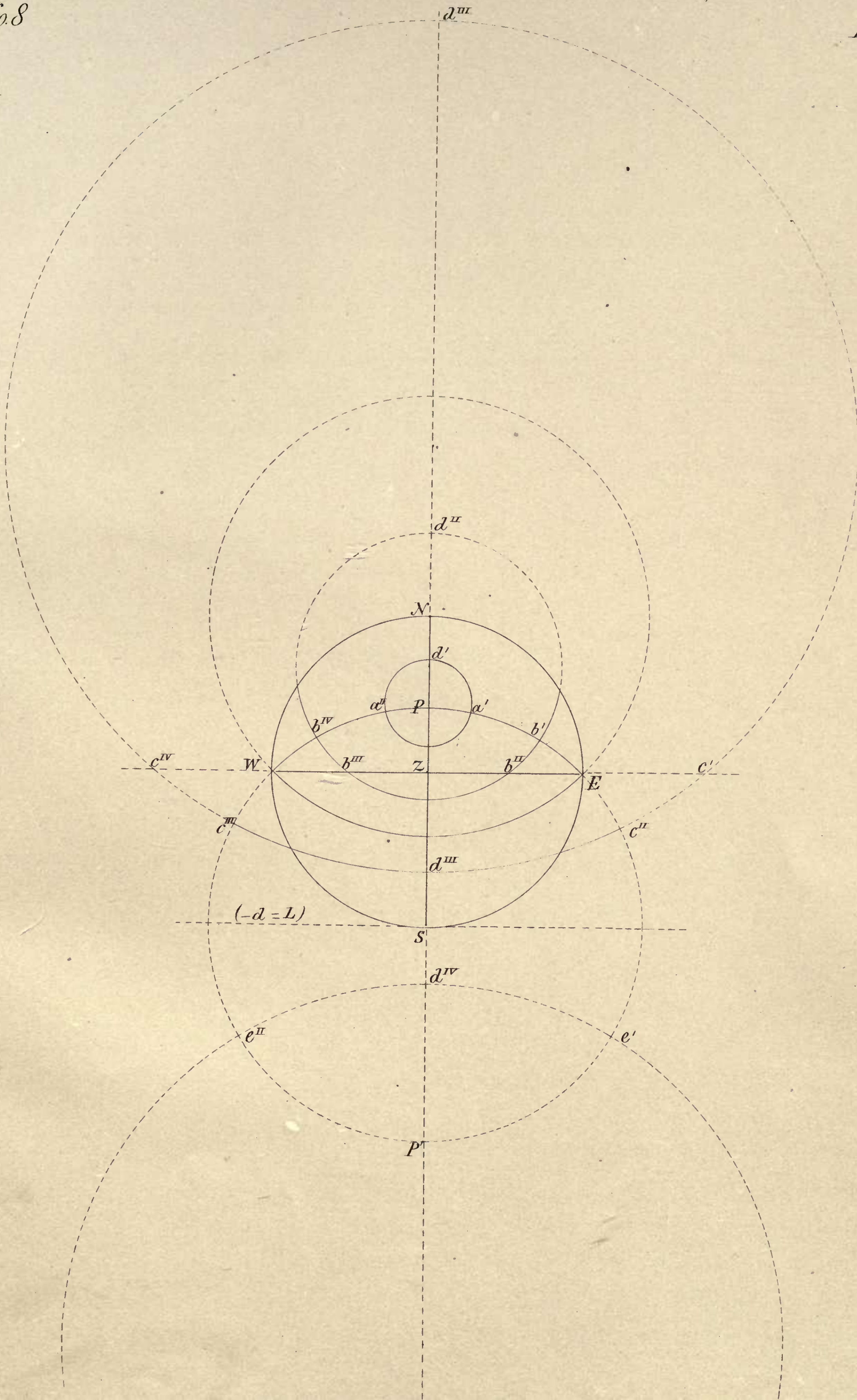
A



Time-Altitude-Azimuth, error in t

No. 8

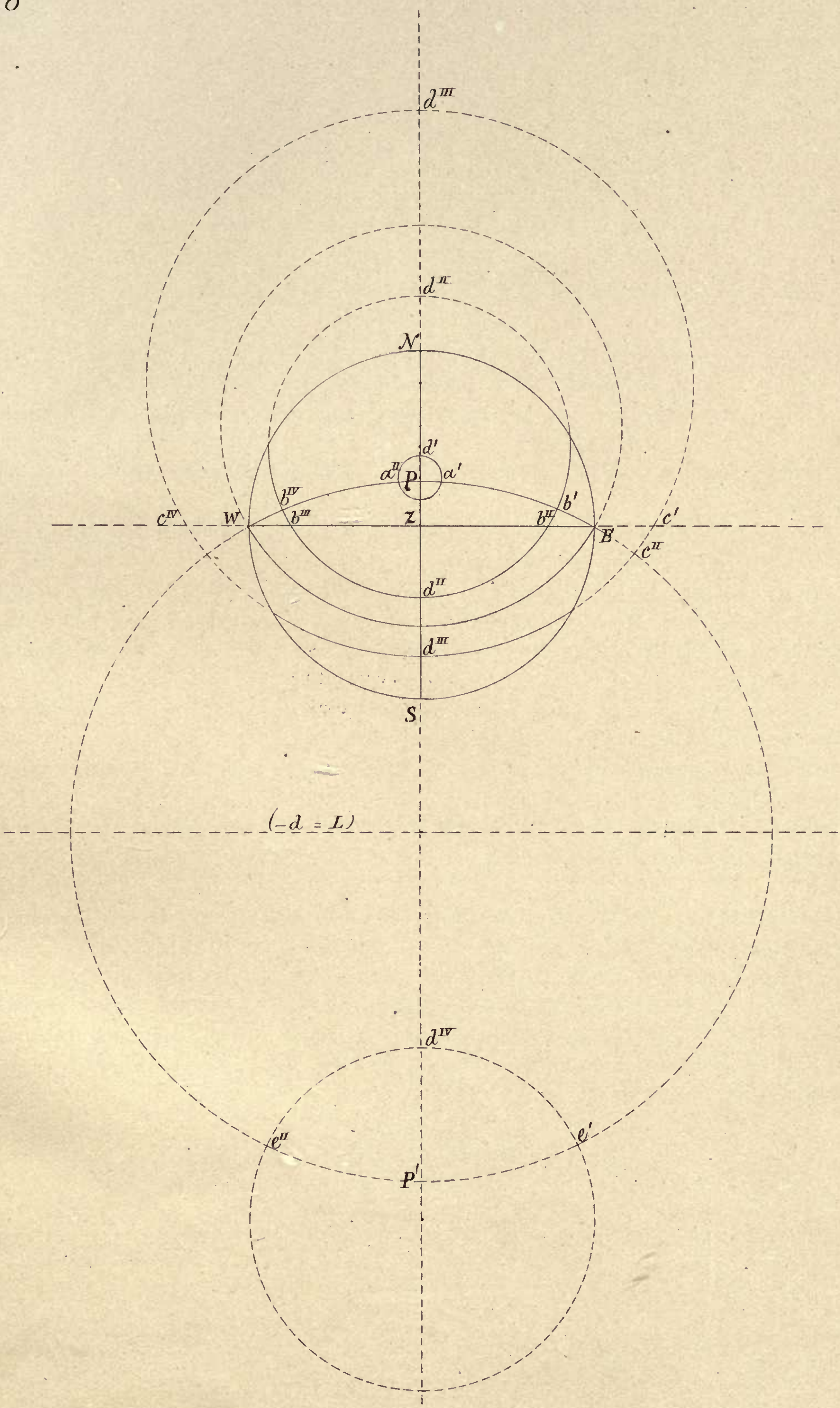
B



Time-Altitude-Azimuth, error in $\underline{t}$

No. 8

C



LOCUS NO. 9.—TIME-ALTITUDE-AZIMUTH ERROR IN d.

[Summary of pp. 28, 48, 49, 81.] $$\frac{dZ}{dd} = - \tan Z \tan d = - \frac{\sin t \sin d}{\cos Z \cos h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

Branches: 1. Absolute min., the meridian NPS.

2. Absolute max., the prime-vertical WZE.

3. Algebraic max. and min., giving num. max., the curve of elongations $Pa'Za''$ and $e'P'e''$.

Alg. max. and min. only for $\pm d > L$; alg. min. west and alg. max. east for $+d$, and conversely for $-d$.

For $+d > L$, at low. cul., d', abs. min. 0; a', alg. max., num. max. (+); at upper transit, d', abs. min. 0; a'', alg. min., num. max. (−).

For $+d = L$, at low. cul., abs. min. 0; at upper cul. indeterminate, but may be called abs. max. for *close to* the zenith the error dZ is very large. Z is the meeting-point of the antipathetic curves, but its vicinity, through which $d = L$ travels, partakes of the character of the absolute max.

For $+d < L$, at d'', low. cul., abs. min. 0; b', abs. max. ∞; d'', upper cul., abs. min. 0; b'', abs. max. ∞. The equator and $-d < L$, on merid., abs. min.; on p. v. abs. max.

For $-d = L$, the converse of $+d = L$; the vicinity of the nadir the unfavorable region, upper cul., abs. min. 0.

For $-d > L$, num. max. east (e') and west (e'') have contrary signs to those for $+d > L$.

Observations east and west on the same parallel of altitude will eliminate the error in dZ.

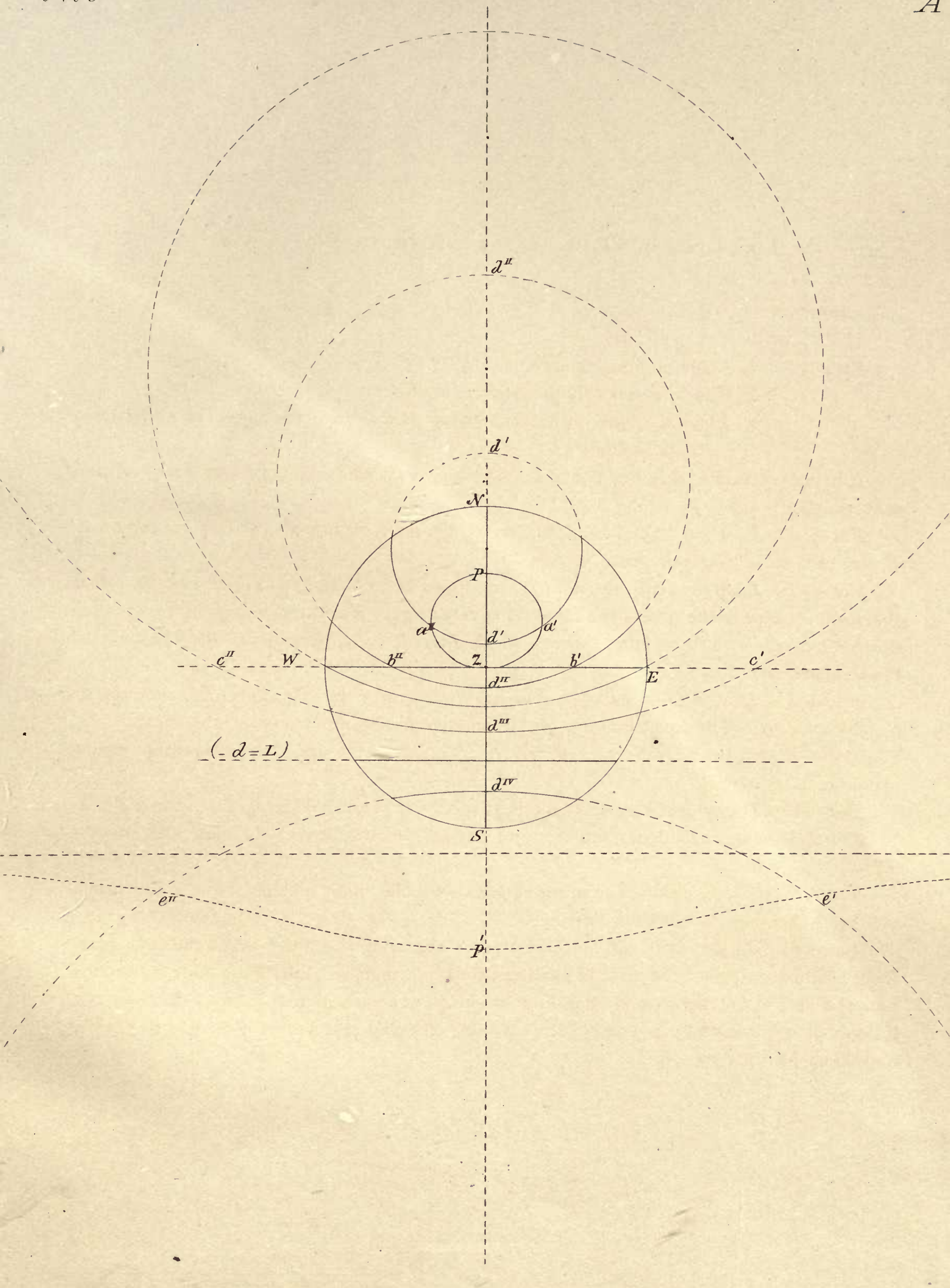
For error in every datum, the prime-vertical gives abs. max., and points near the prime-vertical, in bearing, should be avoided.

Since errors in d and in h will be eliminated by observing east and west on the same parallel of altitude, and since the error in t will not thus be eliminated, but, if the body is observed on *either* side of the meridian, *on* the six-hour circle the error will reduce to 0,—the best condition will be afforded by a close circumpolar star observed east and west as near to the six-hour circle as it can be seized.

No 9

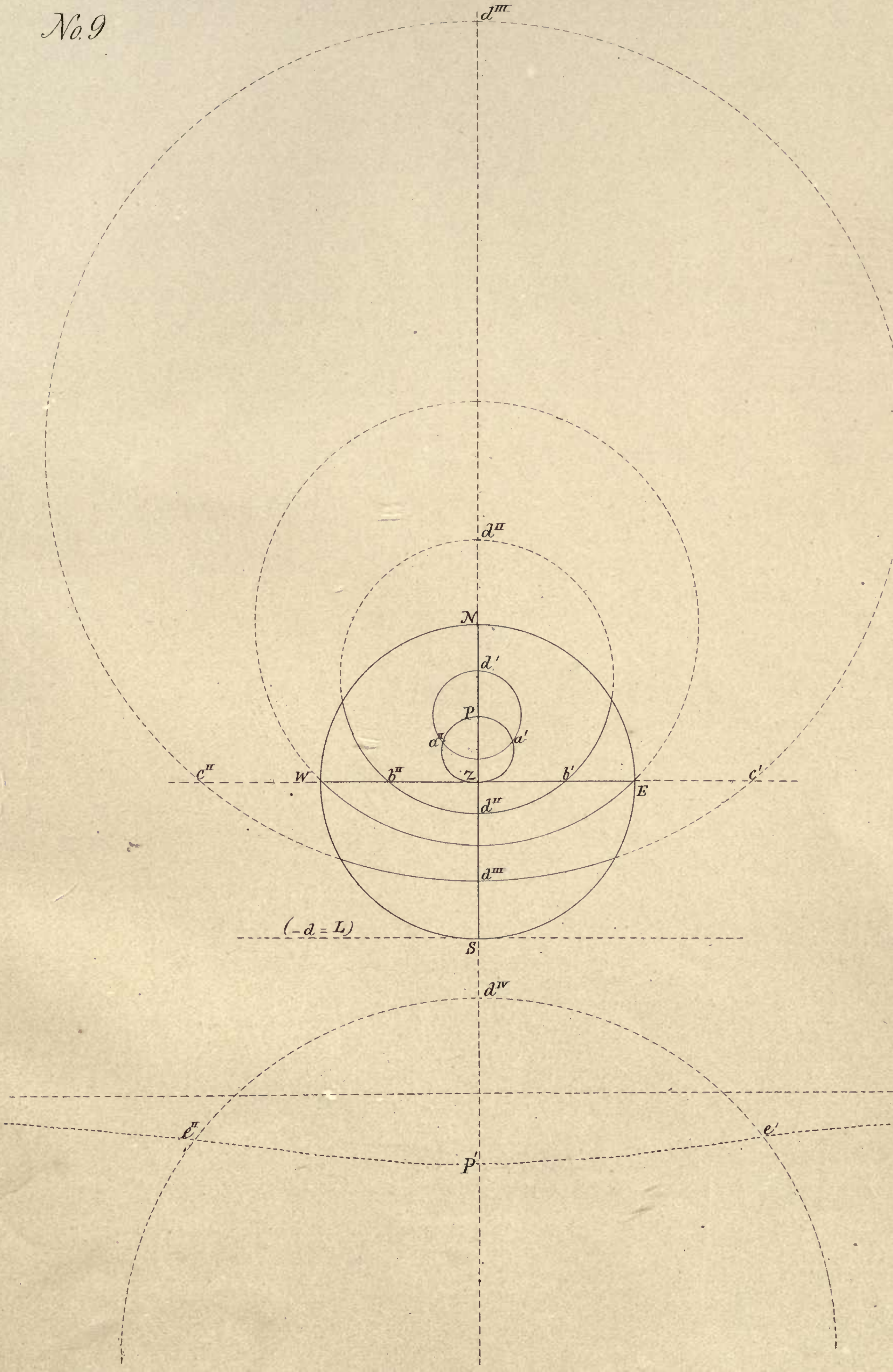
Time-Altitude-Azimuth, error in d

A



Time-Altitude-Azimuth, error in d

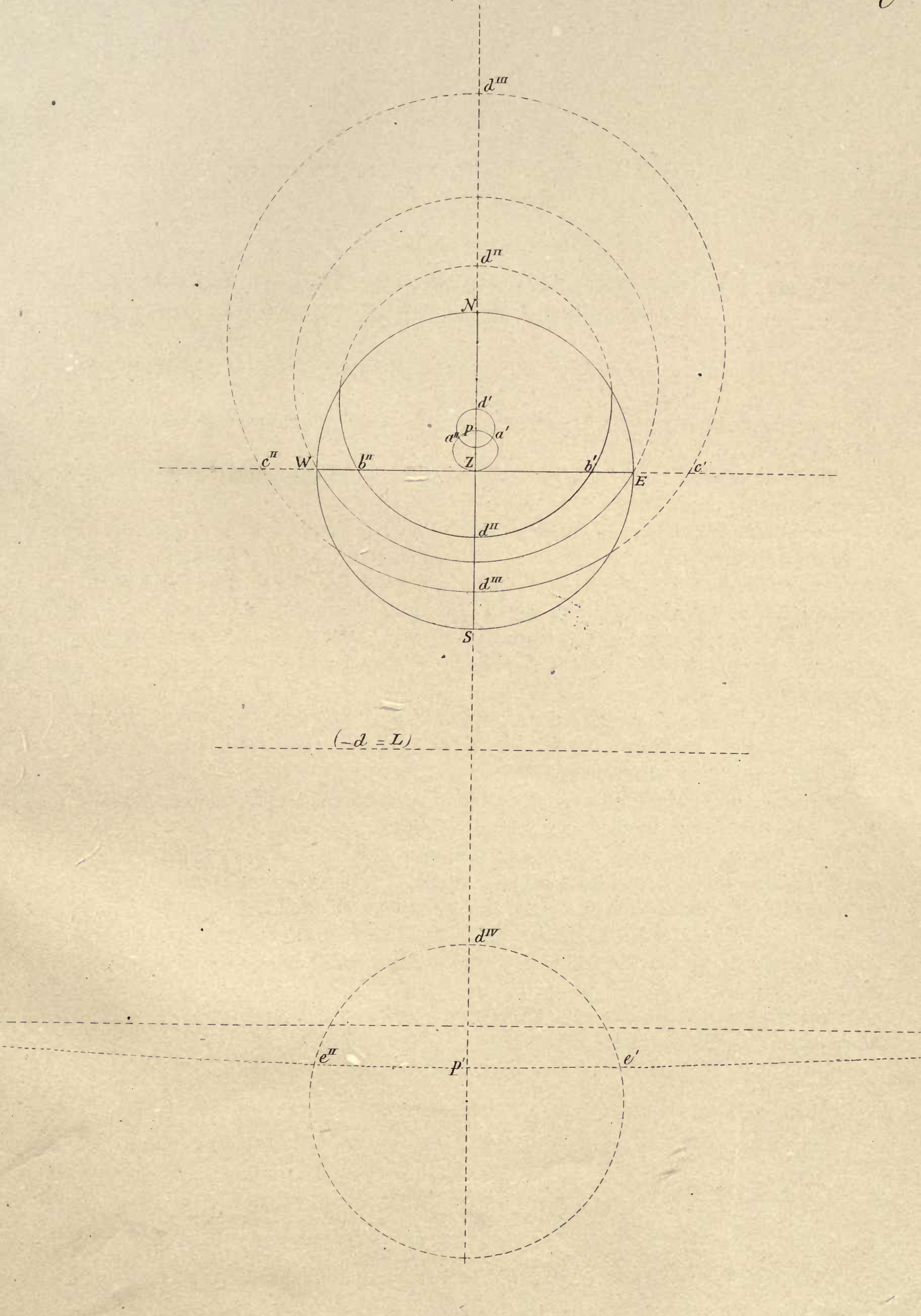
No. 9



Time-Altitude-Azimuth, error in $\underset{\cdot}{d}$

No. 9

C



Locus No. 10.—For Error in L. Time-Azimuth and Altitude-Azimuth equally good.

[Summary of pp. 49–53, 81–83.]

Branches: 1. Identical errors in the resulting azimuths computed by both methods,—the prime-vertical, WZE, and the curve of elongations, $Pa''Za'''$ and $e'P'e^{iv}$.

2. Equal resulting errors, numerical, but having contrary signs,—the curve $Pa'b'Ec''e''P'e'''c'''Wb^{iv}a^{iv}P$.

In the plates, the six-hour circle, instead of being drawn full above the horizon, is a dotted line throughout.

The branches of the locus form the boundaries to the regions within which the one method of finding the azimuth is to be preferred to the other.

The method that is the *more favorable* is:

For $+d > L$, *time*-azimuth from the time of low. cul., until the star arrives at a'; *altitude*-azimuth from a' to a''; *time*-azimuth from a'' to a''', crossing the meridian; *altitude*-azimuth from a''' to a^{iv}; *time*-azimuth from a^{iv}, through lower culmination, to a' again.

For $+d = L$, *time*-azimuth until cutting the curve at a point between a' and b'; thence, passing the zenith to the point between a^{iv} and b^{iv}, the *altitude*-azimuth; thenceforward, to a repetition, *time*-azimuth.

For $+d < L$, from lower culmination until b' is reached, *time*-azimuth; b' to b'', *altitude*-azimuth; b'' to b''', *time*-azimuth; b''' to b^{iv}, *altitude*-azimuth; thenceforward *time*-azimuth during the course of the star to b' again.

For $d = 0$, *time*-azimuth throughout, with the reservation that *altitude*-azimuth is equally good at the points E and W, the error reducing to 0 by each method.

For $-d < L$, from lower culmination to c', *time*-azimuth; c' to c'', *altitude*-azimuth; c'' to c''', *time*-azimuth; c''' to c^{iv}, *altitude*-azimuth; thence to c' again, *time*-azimuth.

For $-d = L$, *time*-azimuth during the time the star is above horizon.

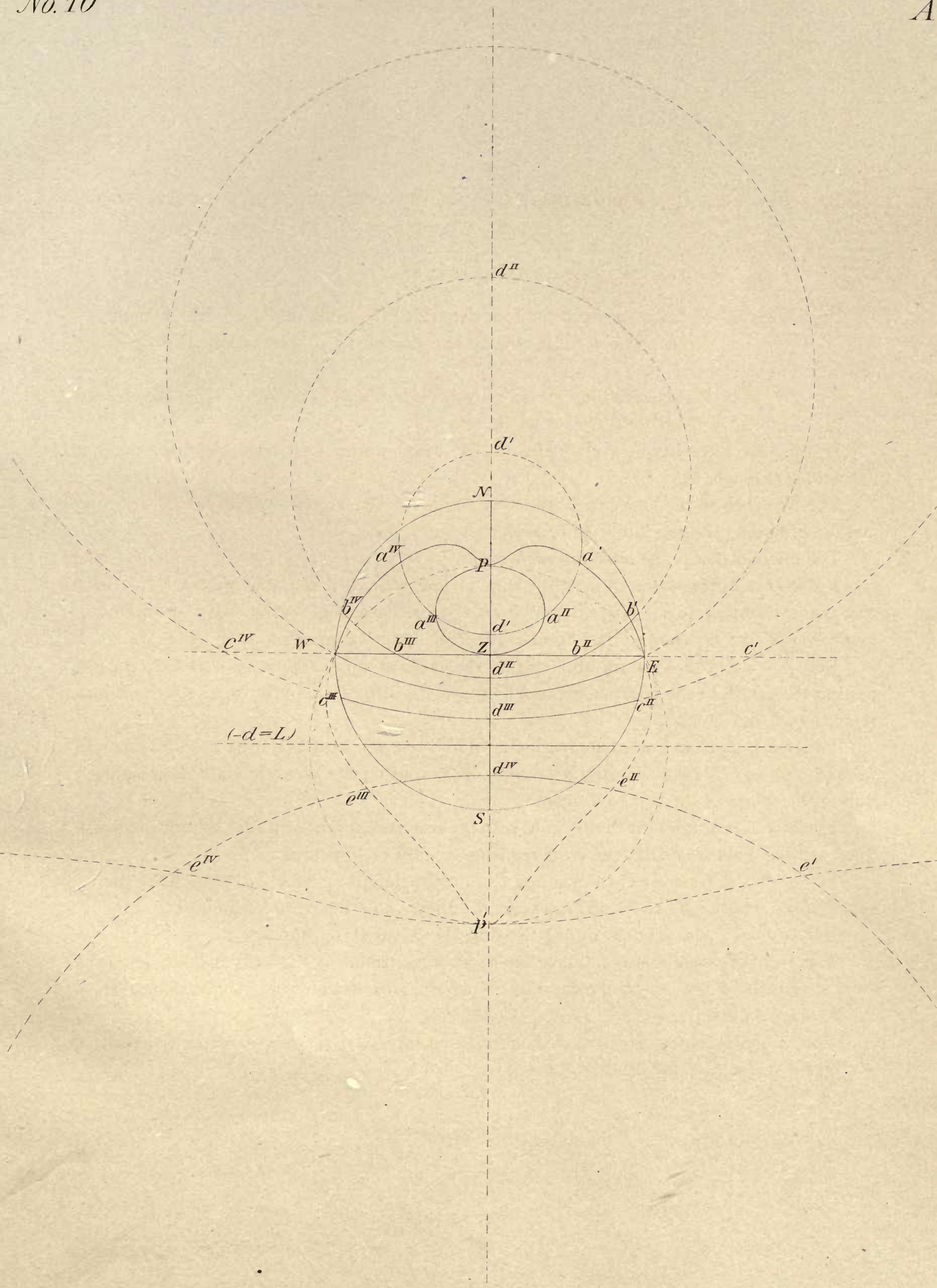
For $-d > L$, from lower culmination to e', *time*-azimuth; e' to e'', *altitude*-azimuth; from e'', passing d^{iv} to e''', *time*-azimuth; e''' to e^{iv}, *altitude*-azimuth; thenceforward to e' again, *time*-azimuth.

For all bodies having negative declinations, and for $d = 0$, the *time-azimuth* is the better during the whole time of visibility of the star.

For error in L, Time-Azimuth and Altitude-Azimuth equally favorable.

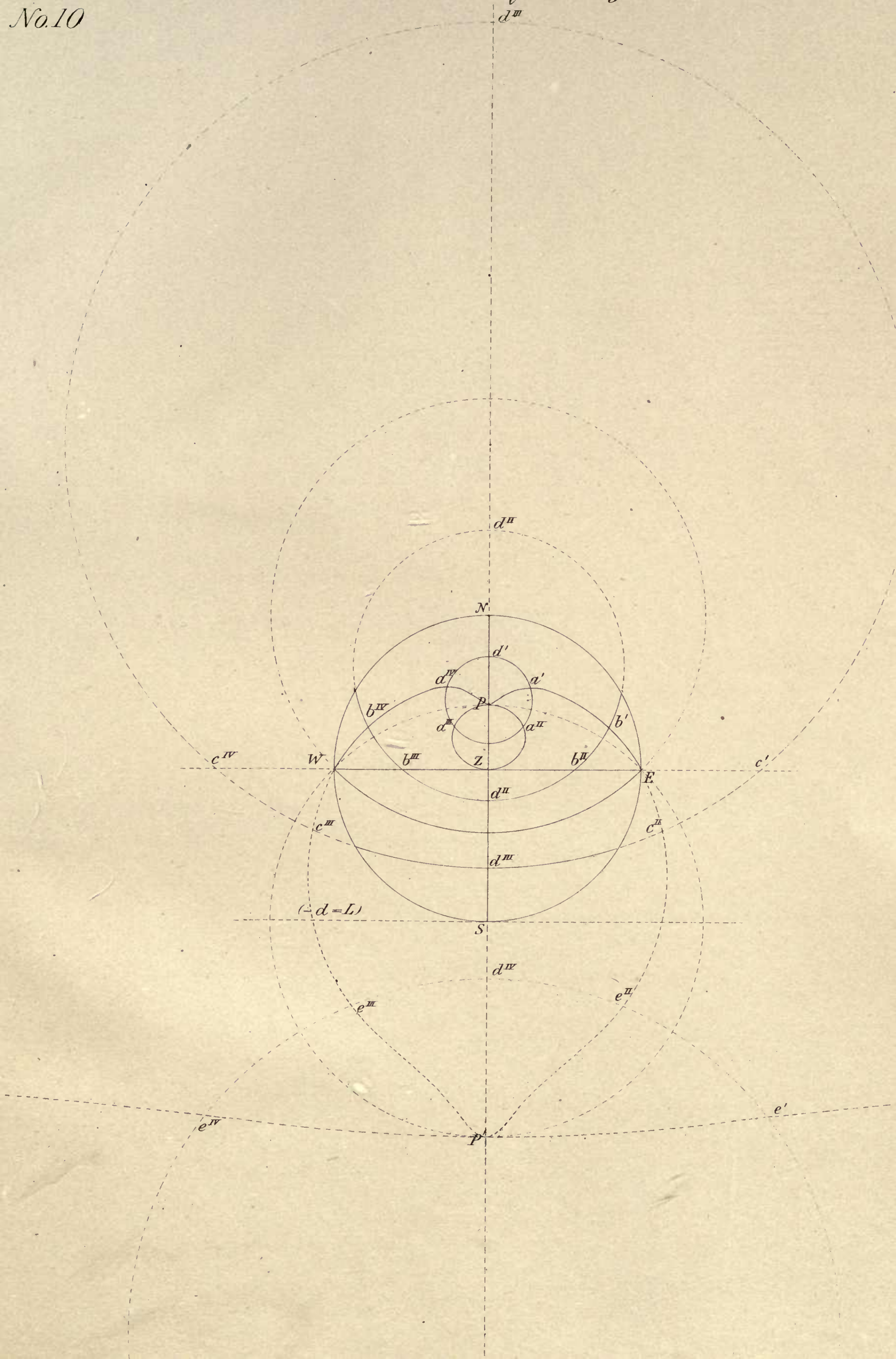
No. 10

A



For error in L, Time-Azimuth and Altitude-Azimuth equally favorable.

No. 10



For error in L, Time-Azimuth and Altitude-Azimuth equally favorable.

No:10